

Exotic Universal Solutions in Cubic Superstring Field Theory

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Abstract

We present a class of analytic solutions of cubic superstring field theory in the universal sector on a non-BPS D-brane. Computation of the action and gauge invariant overlap reveal that the solutions carry half the tension of a non-BPS D-brane. However, the solutions do not satisfy the reality condition. In fact, they display an intriguing topological structure: We find evidence that conjugation of the solutions is equivalent to a gauge transformation that cannot be continuously deformed to the identity.

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1 Introduction

There are two interesting ways to formulate the field equations of an open NS superstring. The first comes from Berkovits' nonpolynomial string field theory[1], and involves a ghost

and picture number 0 string field Φ in the large Hilbert space subject to the equations of motion

$$\eta_0(e^{-\Phi} Q e^{\Phi}) = 0. \quad (1.1)$$

The second comes from cubic superstring field theory[2, 3], and involves a ghost number 1, picture number 0 string field Ψ in the small Hilbert space subject to the equations of motion

$$Q\Psi + \Psi^2 = 0. \quad (1.2)$$

The cubic equations of motion are simpler, in that they are polynomial and directly analogous to the field equations for the open bosonic string[4], but suffer from the disadvantage that they are difficult to derive from a completely reliable action². Nevertheless the Berkovits and cubic equations are known to be perturbatively equivalent[14], and nonperturbatively any Berkovits solution generates a cubic solution via the equation

$$\Psi = e^{-\Phi} Q e^{\Phi}. \quad (1.5)$$

However, the reverse is not true. The existence of a cubic solution Ψ does not guarantee the existence of a Berkovits solution e^{Φ} satisfying (1.5). For example, cubic superstring field theory has a “tachyon vacuum” on a BPS D-brane[15, 16]. There is no evidence for such a solution in Berkovits’ string field theory, either analytically[15, 17] or numerically[18].

In this paper we present a new example of this phenomenon. We show that the cubic equations of motion on a non-BPS D-brane possess an unexpected class of universal solutions which appear not to exist in Berkovits string field theory. The existence of these solutions is highly nontrivial, but their physical interpretation is unknown. They possess a number of surprising properties which may be important for our understanding of string field theory:

²To evaluate the energy in this paper, we will use the action originally proposed by Preitschopf, Thorne, and Yost[2]:

$$S = \frac{1}{2} \langle\langle \Psi Q \Psi \rangle\rangle + \frac{1}{3} \langle\langle \Psi^3 \rangle\rangle. \quad (1.3)$$

The bracket $\langle\langle \cdot \rangle\rangle$ is defined using the Witten vertex with a midpoint insertion

$$Y_{-2} = Y(i) \tilde{Y}(i), \quad Y(z) = -\partial \xi e^{-2\phi} c(z). \quad (1.4)$$

See appendix A. The problems with this action are well-known, including difficulties with the convergence of level truncation[5, 6, 7, 8], complications with gauge fixing and perturbation theory[2], problems with the Ramond sector[9, 10], and the existence of a singular kernel for the bracket[11]. Recently there has been some interest in finding a more suitable action[10, 12, 13], though the success of these proposals remains unclear.

- The solutions are not real. In fact, every solution appears to be related to its conjugate by a topologically nontrivial gauge transformation.
- The solutions appear not to exist in Berkovits’ string field theory.
- If we ignore the reality condition and compute observables, the solutions turn out to carry half the tension of a non-BPS D-brane.

We will call them *half-brane solutions*, in accordance with their tension. The solutions are significant in that they appear to be the first examples of topological solutions in open string field theory. We hope that they can provide a deeper understanding of the topology of the string field algebra, with the ultimate goal of providing a “microscopic” description of D-brane charges in the context of string field theory.

This paper is organized as follows. In section 2 we construct half-brane solutions by extending the wedge algebra to include generators of worldsheet supersymmetry. We attempt an analogous construction in Berkovits string field theory, and show that it fails. In section 3 we prove that half-brane solutions do not satisfy the string field reality condition. We also show, within a controlled subalgebra of states, that every half-brane solution is related to its conjugate by a topologically nontrivial gauge transformation. In section 4 we discuss the regularization and phantom piece for the half-brane solution. The phantom term offers an interesting perspective on the nature of convergence in the wedge algebra, and suggests a more general technique for constructing states in the wedge algebra—including, possibly, projector states distinct from the sliver and identity string field. In section 5 we calculate the action and closed string tadpole. We find highly nontrivial agreement between these observables, indicating that the solutions represent a state with half the tension of a non-BPS D-brane. We end with some discussion.

2 Solution

2.1 Algebra

To begin we need to recall some facts about the algebra of string fields³ on a non-BPS D-brane. The algebra has two \mathbb{Z}_2 gradings: Grassmann parity ϵ , which corresponds to the Grassmann parity of the vertex operator creating the string field; and worldsheet spinor

³In this paper we use the left handed convention for the star product[19]. Other standard sources for the superstring[6, 8, 18] use the right handed convention[20], and there are some important sign differences in the GSO(−) sector. See appendix A.

number F , which tells us whether the field is in the $\text{GSO}(+)$ or $\text{GSO}(-)$ sector. Fields in the algebra are assigned internal Chan-Paton factors according to the table:

ϵ	F	CP factor
0	0	\mathbb{I}
1	0	σ_3
0	1	σ_2
1	1	σ_1

The BRST charge Q and the midpoint insertion Y_{-2} both implicitly carry an internal CP factor of σ_3 , and the 1-string vertex $\langle\langle \cdot \rangle\rangle$ automatically contains a factor of $1/2$ times the trace over internal CP matrices. To keep track of signs when commuting vertex operators and CP factors past each other, it is helpful to define what we will call *effective Grassmann parity*

$$E = \epsilon + F \pmod{2}. \quad (2.1)$$

In particular, the star algebra has a natural graded commutator⁴

$$[\![\Psi, \Phi]\!] = \Psi\Phi - (-1)^{E(\Psi)E(\Phi)+F(\Psi)F(\Phi)}\Phi\Psi, \quad (2.2)$$

where Ψ, Φ implicitly carry the appropriate CP factor. This suggests that the star product on a non-BPS brane has a structure analogous to a product of matrices whose entries contain two mutually commuting types of Grassmann number, the first has a Grassmannality measured by E and the second by F . However, only effective Grassmann parity enters into the string field theory axioms:

$$\begin{aligned} Q(\Psi\Phi) &= (Q\Psi)\Phi + (-1)^{E(\Psi)}\Psi(Q\Phi), \\ \langle\langle \Psi\Phi \rangle\rangle &= (-1)^{E(\Psi)E(\Phi)}\langle\langle \Phi\Psi \rangle\rangle. \end{aligned} \quad (2.3)$$

In particular, the physical string field Ψ on a non-BPS D-brane must be *effective* Grassmann odd. For example, the tachyon vertex operator $\gamma(0)$ is Grassmann even in the traditional sense, but since it carries worldsheet spinor number, it counts as “effectively” Grassmann odd.

⁴This “double bracket” commutator should be distinguished from the graded commutator $[\Psi, \Phi] = \Psi\Phi - (-1)^{E(\Psi)E(\Phi)}\Phi\Psi$ which emerges naturally from the action, both in the infinitesimal gauge transformation and the kinetic operator around a nontrivial solution. The single bracket $[\cdot, \cdot]$ is only graded according to effective Grassmann parity.

	ghost number	effective Grassmann parity	worldsheet spinor number	scaling dimension	reality	twist
K	0	0	0	1	real	1
B	-1	1	0	1	real	1
c	1	1	0	-1	real	-1
G	0	0	1	$\frac{1}{2}$	real	$-i$
γ	1	1	1	$-\frac{1}{2}$	real	$-i$

Table 1: Some important quantum numbers for the atomic fields. Scaling dimension refers to the eigenvalue of the field under the action of the operator $\frac{1}{2}\mathcal{L}^- = \frac{1}{2}(\mathcal{L}_0 - \mathcal{L}_0^*)$. Reality and twist refer to the eigenvalues of the fields under reality and twist conjugation, defined in appendix A. By “real” we mean that the fields have eigenvalue 1 under reality conjugation.

With these preparations we are ready to give the algebraic setup for our solution. The solution is constructed by taking star products of four “atomic” string fields:

$$K, \quad B, \quad c, \quad G. \quad (2.4)$$

The ghost number, effective Grassmann parity, and some other important quantum numbers of these fields are summarized in table 1. We can construct K, B, c, G by acting certain operators on the identity string field $|I\rangle$:

$$\begin{aligned} K &= \mathbb{I} \otimes \mathcal{L}_L^+ |I\rangle, & B &= \sigma_3 \otimes \mathcal{B}_L^+ |I\rangle, \\ c &= \sigma_3 \otimes \frac{1}{\pi} c(1) |I\rangle, & G &= \sigma_1 \otimes \mathcal{G}_L |I\rangle. \end{aligned} \quad (2.5)$$

The subscript L above denotes taking the left half of the charges:

$$\begin{aligned} \mathcal{L}^+ &= \mathcal{L}_0 + \mathcal{L}_0^*, & \mathcal{L}_0 &= f_S^{-1} \circ L_0, \\ \mathcal{B}^+ &= \mathcal{B}_0 + \mathcal{B}_0^*, & \mathcal{B}_0 &= f_S^{-1} \circ b_0, \\ \mathcal{G} &= f_S^{-1} \circ G_{-1/2}, \end{aligned} \quad (2.6)$$

where $f_S^{-1}(z) = \tan \frac{\pi}{2} z$ is the inverse of the sliver conformal map[21, 22] and the star * denotes BPZ conjugation. Another definition of these fields is given by mapping them to

operator insertions inside correlation functions on the cylinder:

$$\begin{aligned} K &\rightarrow \mathbb{I} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T(z), & B &\rightarrow \sigma_3 \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} b(z), \\ c &\rightarrow \sigma_3 c(z), & G &\rightarrow \sigma_1 \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} G(z). \end{aligned} \quad (2.7)$$

See [20, 23] and the appendix of [19] for an explanation of how this mapping works. The essentially new ingredient in our algebraic setup is the string field G . It lives in the $\text{GSO}(-)$ sector, and corresponds to a line integral insertion of the worldsheet supercurrent $G(z)$. Note that to define G we need to “split” the operator \mathcal{G} into left and right halves. Such splittings are potentially anomalous[24], but in this case the splitting appears to be regular (see appendix B for more details).

The fields K, B, c, G freely generate a subalgebra of the open string star algebra subject to the relations

$$\begin{aligned} G^2 &= K, & Bc + cB &= 1, & B^2 &= c^2 = 0, \\ K, B, G &\text{mutually commute.} \end{aligned} \quad (2.8)$$

K generates the algebra of wedge states[20, 25] in the sense that any star-algebra power of the $SL(2, \mathbb{R})$ vacuum $\Omega = |0\rangle$ can be written $\Omega^\alpha = e^{-\alpha K}$. It is useful to define the operators:

$$\partial = [K, \cdot], \quad \delta = \llbracket G, \cdot \rrbracket. \quad (2.9)$$

In the cylinder coordinate frame, ∂ generates an infinitesimal worldsheet translation and δ generates a worldsheet supersymmetry variation. They are derivations of the star product:

$$\begin{aligned} \partial(\Psi\Phi) &= (\partial\Psi)\Phi + \Psi(\partial\Phi), \\ \delta(\Psi\Phi) &= (\delta\Psi)\Phi + (-1)^{F(\Psi)}\Psi(\delta\Phi). \end{aligned} \quad (2.10)$$

Since the supersymmetry variation of c produces the γ ghost, it is helpful to introduce the corresponding string field:

$$\gamma = \sigma_2 \otimes \frac{1}{\sqrt{\pi}} \gamma(1)|I\rangle \rightarrow \sigma_2 \gamma(z) = \sigma_2 \eta e^\phi(z). \quad (2.11)$$

Then we have

$$\begin{aligned} \delta c &= 2i\gamma, & \delta\gamma &= -\frac{i}{2}\partial c, \\ \delta G &= 2K, & \delta K &= 0, & \delta B &= 0. \end{aligned} \quad (2.12)$$

Note that δ satisfies the supersymmetry algebra $\delta^2 = \partial$. Since K is the worldsheet superpartner of G , together these fields generate a supersymmetric extension of the wedge algebra, which we will call the *wedge superalgebra*.

The algebra generated by K, B, c, G is closed under the action of the BRST operator

$$QK = 0, \quad QB = K, \quad Qc = cKc - \gamma^2, \quad QG = 0. \quad (2.13)$$

Therefore it makes sense to look for solutions to the cubic equations of motion

$$Q\Psi + \Psi^2 = 0 \quad (2.14)$$

within this subalgebra.

2.2 Half-Brane Solutions

In this paper we study solutions of the form

$$\Psi[f] = \left(c \frac{KB}{1-f} c + B\gamma^2 \right) f, \quad (2.15)$$

where f is a string field in the wedge superalgebra. Multiplying and dividing by \sqrt{f} gives a gauge equivalent solution

$$\hat{\Psi}[f] = \sqrt{f} \left(c \frac{KB}{1-f} c + B\gamma^2 \right) \sqrt{f}, \quad (2.16)$$

which is twist symmetric⁵. These are exactly the same formal expressions which give the pure gauge and tachyon vacuum solutions of [15]. The only new ingredient here is f , which can depend on G . Explicitly,

$$f = f_+ + Gf_-, \quad (2.17)$$

where $f_{\pm} = f_{\pm}(K)$ are functions of K only. In terms of f_{\pm} the solution takes the form

$$\Psi[f] = \left(cKB \frac{1 - f_+ + Gf_-}{(1 - f_+)^2 - Kf_-^2} c + B\gamma^2 \right) (f_+ + Gf_-). \quad (2.18)$$

With a little extra work we can also compute $\sqrt{f_+ + Gf_-}$ to find the twist symmetric solution.

⁵By twist symmetric, we mean that the GSO(+) and GSO(-) components of the solution separately have definite eigenvalue under twist conjugation. In particular, the GSO(+) component is made of states whose $L_0 + 1$ eigenvalues are even integers, and the GSO(-) component is made of states whose $L_0 + \frac{1}{2}$ eigenvalues are odd integers.

The physical interpretation of these solutions depends on the choice of f_{\pm} . To see how, it is helpful to employ a formal analysis in the \mathcal{L}^- level expansion, which is an easy and apparently reliable method for identifying gauge orbits in solutions of this type[19]. Recall that $\mathcal{L}^- = \mathcal{L}_0 - \mathcal{L}_0^*$ is a reparameterization generator and a derivation. This means that the star product of two \mathcal{L}^- eigenstates is itself an eigenstate, and the eigenvalues add. Since K, B, c, G are eigenstates of \mathcal{L}^- (see table 1), we can find the \mathcal{L}^- level expansion of $\Psi[f]$ by expanding in powers of K and ordering the terms in sequence of increasing scaling dimension. The expansion can take one of three different forms, depending on the behavior of f_{\pm} at $K = 0$:

$$\text{Pure Gauge : } f_+(0) \neq 1, \quad (2.19)$$

$$\text{Half Brane : } f_+(0) = 1, \quad f_-(0) \neq 0, \quad (2.20)$$

$$\text{Tachyon Vacuum : } f_+(0) = 1, \quad f_-(0) = 0, \quad f'_+(0) \neq 0, \quad (2.21)$$

corresponding to the expansions

$$\begin{aligned} \text{Pure Gauge : } \Psi &= \frac{f_+(0)}{1 - f_+(0)} Q(Bc) - \frac{f_+(0)^2}{1 - f_+(0)} B\gamma^2 + \dots, \\ \text{Half Brane : } \Psi &= -\frac{1}{f_-(0)} cGBc + \dots, \\ \text{Tachyon Vacuum : } \Psi &= -\frac{1}{f'_+(0)} c + \dots, \end{aligned} \quad (2.22)$$

where ... denotes higher level terms. Each expansion formally corresponds to a physically distinct gauge orbit⁶ within our general ansatz (see figure 2.1). The pure gauge and tachyon vacuum solutions are known[15], but the so-called *half-brane solutions* are new. These are the main subject of this paper.

Often one can get insight into the physics of a solution by inspecting its leading term in the \mathcal{L}^- level expansion. For the tachyon vacuum the leading term is proportional to the c ghost, which is responsible for the absence of cohomology at the vacuum[15]. For pure gauge solutions, the leading term is BRST exact to linear order, corresponding to the fact that pure gauge solutions represent a deformation of the perturbative vacuum by a trivial element of the BRST cohomology. For half brane solutions, the full meaning

⁶Following [19], one can construct a formal gauge transformation relating solutions with different choices of f : $\Psi[f'] = g^{-1}(Q + \Psi[f])g$. However, the gauge transformation breaks down if f and f' do not share the same boundary conditions at $K = 0$, (2.19)-(2.21), since either g or g^{-1} would formally require inverse powers of K in its \mathcal{L}^- level expansion. Inverse powers of K are not constructible states within the wedge algebra.

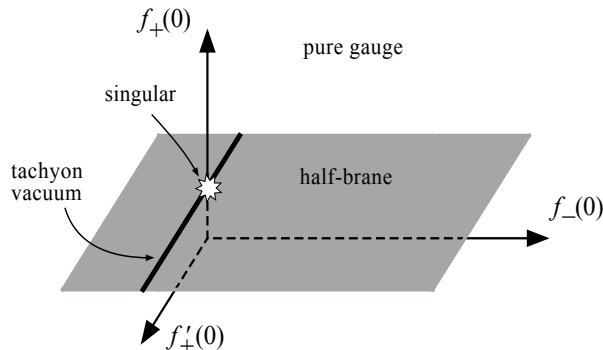


Figure 2.1: Three dimensional “phase space” of solutions, parameterized by $f_+(0)$, $f_-(0)$ and $f'_+(0)$. Tachyon vacuum solutions sit on a line embedded in a plane of half-brane solutions, which themselves are embedded in an ambient space of pure gauge solutions. Note that the point $f_+(0) = 1, f_-(0) = 0, f'_+(0) = 0$ appears to represent singular solutions.

of the leading term $cGBc$ is not clear to us. However, it is worth noting that $cGBc$ has twist eigenvalue $+i$:

$$(cGBc)^{\S} = +i cGBc. \quad (2.23)$$

Therefore half-brane solutions result from condensation of states in the $\text{GSO}(-)$ sector carrying odd integer eigenvalues of $L_0 + \frac{1}{2}$. This is peculiar since all of these states carry positive mass squared. The more familiar states responsible for tachyon condensation carry even integer $L_0 + \frac{1}{2}$, and in fact these states have vanishing expectation value in the twist even solution (2.16). The fact that half-brane solutions result from “condensation” of massive modes of the open string is one way to anticipate that they cannot satisfy the reality condition.

Let us give two explicit examples of half-brane solutions. The first comes by setting

$$f = f_+ + Gf_- = \frac{1}{1 - iG}, \quad (2.24)$$

and takes the form

$$\Psi_{\text{simp}} = \left[icGBc + Q(Bc) \right] \frac{1 + iG}{1 + K}. \quad (2.25)$$

We will explain the factor of i shortly. We will call this the *simple* half-brane solution, since it is in many ways analogous to the “simple” tachyon vacuum introduced in [19]. In particular, (2.25) requires no phantom term, and gives the most straightforward calculation of the action and gauge invariant overlap. Another solution, which is likely to be

better behaved in the level expansion (see appendix F and [19]), comes from setting

$$f = f_+ + Gf_- = (1 + iaG)\Omega, \quad (2.26)$$

where $a \neq 0$ is a parameter. It takes the form,

$$\Psi_{\text{Sch}} = \left[c \frac{KB(1 - \Omega + iaG\Omega)}{(1 - \Omega)^2 + a^2 K \Omega^2} c + B\gamma^2 \right] (1 + iaG)\Omega. \quad (2.27)$$

Unlike (2.25), this solution is composed of wedge states whose angles have strictly positive lower bound. We will call it the *Schnabl-like* solution. To compute the action or gauge invariant overlap, we should express the solution as a regularized sum subtracted against a phantom term. We will explain how to do this in section 4.

2.3 Half-Brane Solutions in Berkovits' String Field Theory

We would now like to know whether half-brane solutions exist in Berkovits' string field theory. The task is to find a pair of string fields (g, g^{-1}) at ghost and picture number zero, and in the large Hilbert space, satisfying

$$Qg = g\Psi, \quad (2.28)$$

$$g^{-1}g = gg^{-1} = 1. \quad (2.29)$$

where Ψ is a cubic half-brane solution. Within a certain subalgebra of states, we will show that these equations have no solutions for g and g^{-1} . A similar approach can be used to argue that Berkovits' string field theory does not have a tachyon vacuum solution on a BPS D-brane[17].

To solve equations (2.28) and (2.29), we must extend our subalgebra to include fields in the large Hilbert space. The minimal and most natural extension is to include the string field

$$A = -\sigma_3 \otimes \xi \partial \xi e^{-2\phi} c(1) |I\rangle \quad (2.30)$$

which satisfies

$$QA = 1, \quad A\gamma^2 = -c, \quad Ac = cA = 0, \quad \llbracket \gamma, A \rrbracket = 0, \quad \llbracket \partial c, A \rrbracket = 0. \quad (2.31)$$

A describes an insertion of an inverse picture changing operator multiplied by the ξ zero mode. It has ghost number -1 , is effective Grassmann odd, carries even worldsheet

spinor number, and has scaling dimension 0. Naively, the field A is enough to generate any Berkovits solution given any cubic solution. To see how, note that

$$g = 1 + A\Psi. \quad (2.32)$$

solves (2.28)[14, 26]. Then, we can *almost* solve (2.29) by expressing g^{-1} as an infinite geometric series in powers of $-A\Psi$. However, this series is not guaranteed to converge. This is why the cubic and Berkovits equations of motion are not *a priori* equivalent.

We search for a Berkovits half-brane by making the most general possible ansatz in the subalgebra generated by K, B, c, G and A . Expand Ψ and (g, g^{-1}) into \mathcal{L}^- eigenstates as follows:

$$\begin{aligned} \Psi &= \Psi_{-1/2} + \Psi_0 + \Psi_{1/2} + \dots, \\ g &= g_{-1/2} + g_0 + g_{1/2} + \dots, \\ g^{-1} &= \bar{g}_{-1/2} + \bar{g}_0 + \bar{g}_{1/2} + \dots, \end{aligned} \quad (2.33)$$

where the subscripts refers to the $\frac{1}{2}\mathcal{L}^-$ eigenvalue of the fields. If Ψ is a cubic half-brane solution, its expansion takes the general form

$$\begin{aligned} \Psi_{-1/2} &= -\frac{1}{\alpha_1 + \alpha_2} cGBc, \\ \Psi_0 &= -\frac{\alpha_1}{\alpha_1 + \alpha_2} GcGBc - \frac{\alpha_2}{\alpha_1 + \alpha_2} cGBcG + \frac{\beta_1 + \beta_2}{(\alpha_1 + \alpha_2)^2} cKBc + B\gamma^2, \\ \Psi_{1/2} &= \dots, \end{aligned} \quad (2.34)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants parameterizing the gauge orbit up to this level. The most general ansatz for g is

$$\begin{aligned} g_{-1/2} &= x A\gamma, \\ g_0 &= y_1 + y_2 Bc + y_3 A\partial c + y_4 GA\gamma + y_5 A\gamma G, \\ g_{1/2} &= \dots, \end{aligned} \quad (2.35)$$

where x and y_1, \dots, y_5 are coefficients to be determined by solving the equations of motion. We make a similar ansatz for g^{-1} . Now plug these formulas into (2.28) and solve level by level:

$$\begin{aligned} 0 &= g_{-1/2}\Psi_{-1/2}, \\ Qg_{-1/2} &= g_{-1/2}\Psi_0 + g_0\Psi_{-1/2}, \\ &\vdots \end{aligned} \quad (2.36)$$

The lowest level equation is trivially satisfied. Plugging (2.34) and (2.35) into the next equation gives

$$xQ(A\gamma) = \frac{2i\alpha_1 x - y_1 + 2iy_5}{\alpha_1 + \alpha_2} cGBc + x\gamma cB. \quad (2.37)$$

Note that the right hand side is in the small Hilbert space. Acting with η_0 therefore gives

$$xQ(\eta_0(A\gamma)) = 0 \quad (2.38)$$

The field $\eta_0(A\gamma)$ is the zero momentum tachyon in the -1 picture. Since the zero momentum tachyon is off-shell, (2.38) implies that the coefficient x vanishes, i.e. $g_{-1/2} = 0$. A similar argument also shows that $\bar{g}_{-1/2} = 0$. Equation (2.36) then implies that g_0 has a right kernel:

$$g_0\Psi_{-1/2} = 0. \quad (2.39)$$

To construct g^{-1} , we must solve (2.29) level by level:

$$\begin{aligned} \bar{g}_{-1/2}g_{-1/2} &= 0, \\ \bar{g}_{-1/2}g_0 + \bar{g}_0g_{-1/2} &= 0, \\ \bar{g}_{-1/2}g_{1/2} + \bar{g}_0g_0 + \bar{g}_{1/2}g_{-1/2} &= 1, \\ &\vdots \end{aligned} \quad (2.40)$$

Since $g_{-1/2} = \bar{g}_{-1/2} = 0$ this implies

$$\bar{g}_0g_0 = 1, \quad (2.41)$$

but this contradicts the fact that g_0 has a right kernel. Therefore, the Berkovits half-brane solution does not exist in the K, B, c, G, A subalgebra. While it is possible that a more general ansatz is necessary, we believe that this subalgebra is rich enough to capture a half-brane solution, if one were to exist.⁷

3 Reality Condition

Physical solutions in cubic superstring field theory are expected to satisfy the reality condition⁸[6, 27]

$$\Psi^\dagger = \Psi, \quad (3.1)$$

⁷Fuchs and Kroyter suggest[14] a general mapping between cubic and Berkovits solutions $g = 1 + \tilde{A}\Psi$, where \tilde{A} is a midpoint insertion of $\xi\partial\xi e^{-2\phi}c$. However, this solution appears to be too singular to allow for a computation of the Berkovits action.

⁸The dagger (†) refers to a composition of Hermitian and BPZ conjugation. See appendix A. Note that this form of the reality condition is correct only for the left-handed star product convention.

It is important to ask whether half-brane solutions meet this criterion. Surprisingly, the answer is no, according to the following theorem:

Theorem 1. *Under assumptions 1)-4) stated below, there are no half-brane solutions in the K, B, c, G subalgebra satisfying the reality condition.*

Proof. Every solution Ψ in the K, B, c, G subalgebra is associated with a pair of states in wedge algebra:

$$f_+(K), \quad f_-(K). \quad (3.2)$$

We can reconstruct f_+ and f_- from the solution by solving the equations⁹

$$B\Psi B = B \frac{K(f_+ + Gf_-)}{1 - (f_+ + Gf_-)}. \quad \llbracket \beta, \llbracket \beta, \Psi \rrbracket \rrbracket = B(f_+ + Gf_-), \quad (3.3)$$

To prove the theorem, we show that the reality condition is inconsistent with certain regularity conditions which must be imposed on $f_+(K)$ and $f_-(K)$. The regularity conditions are:

- 1) $f_+(0) = 1$ and $f_-(0) \neq 0$ and in particular $f'_+(0)$ is finite.
- 2) $\lim_{K \rightarrow \infty} f_+(K) = 0$ and $\lim_{K \rightarrow \infty} \sqrt{K} f_-(K) = 0$.
- 3) f_+ and f_- are continuous functions of K for all $K \geq 0$.
- 4) The field $\frac{K}{(1-f_+)^2 - Kf_-^2}$, is a continuous function of K for all $K \geq 0$.

Condition 1) is essentially the definition of the half-brane solution. Condition 2) ensures that the solution is not too “identity-like,” so that it can have well-defined action and gauge invariant overlap. Conditions 3) and 4) are motivated by a conjecture due to Rastelli[28] suggesting that the algebra of wedge states should be identified with the C^* -algebra of bounded, continuous functions on the positive real line¹⁰. In particular, 3) and 4) assume that f_+ , f_- and $\frac{K}{(1-f_+)^2 - Kf_-^2}$ must separately be well-defined states in order for the solution itself to be well-defined. Since these fields can be extracted directly from the solution via equation (3.3), this assumption appears necessary.

⁹The most general solution in the K, B, c, G subalgebra can be found by making the most general (formal) pure gauge ansatz, following Okawa[20]. In equation (3.3) β represents a line integral insertion of the β ghost in the sliver coordinate frame.

¹⁰The definition of the algebra of wedge states is not known, but discontinuous functions of K appear to be problematic in the level expansion. For related discussions, see [29].

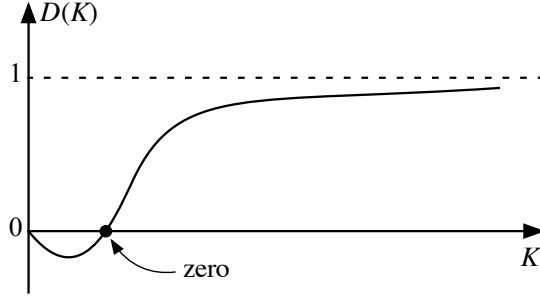


Figure 3.1: If $f_{\pm}(K)$ are real, boundary conditions for the half-brane solution at $K = 0$ and ∞ require that the denominator of the solution (2.18) has a zero for positive K .

The reality condition implies that f_+ and f_- are real functions of K . To see why this contradicts regularity, consider the denominator of the expression appearing in **4)**, which we will call $D(K)$:

$$D(K) = (1 - f_+)^2 - K f_-^2. \quad (3.4)$$

By assumption **1)** we have

$$D(0) = 0 \quad (3.5)$$

and

$$D'(0) = -f_-^2(0). \quad (3.6)$$

Since the slope is negative, we have

$$D(K) < 0 \text{ for some positive } K. \quad (3.7)$$

Now by assumption **2)**

$$\lim_{K \rightarrow \infty} D(K) = 1. \quad (3.8)$$

Since D is continuous by assumption **3)**, this means

$$D(K) = 0 \text{ for some strictly positive } K. \quad (3.9)$$

See figure 3.1. Since $D(K)$ has a zero, the ratio K/D cannot be continuous for all $K \geq 0$ which violates assumption **4)**. \square

It is helpful to see why real f_+ and f_- are problematic in specific examples. Suppose we defined the simple solution in (2.25) without the factor of i :

$$f = f_+ + G f_- = \frac{1}{1 - G}, \quad (3.10)$$

In this case condition **4)** is satisfied since

$$\frac{K}{(1 - f_+)^2 - K f_-^2} = K - 1 \quad (3.11)$$

is a continuous function of K . But condition **3)** is not satisfied: both f_+ and f_- are equal to $\frac{1}{1-K}$, which has a pole at $K = 1$. One could try to define $\frac{1}{1-K}$ using the Schwinger parameterization[19]

$$\frac{1}{1 - K} = - \int_0^\infty dt e^t \Omega^t, \quad (3.12)$$

but since the wedge state Ω^t approaches a constant (the sliver) for large t , this integral diverges exponentially. A second example is the Schnabl-like solution with $a = -i$, so that the factor of i cancels in (2.26). In this case

$$f_+ = f_- = \Omega \quad (3.13)$$

are both real and satisfy **3)**, but

$$\frac{K}{(1 - f_+)^2 - K f_-^2} = \frac{K}{(1 - \Omega)^2 - K \Omega^2} \quad (3.14)$$

has a pole at $K \approx 0.931$ and violates **4)**. One can try to define this state by a geometric series

$$\frac{K(1 - \Omega)}{(1 - \Omega)^2 - K \Omega^2} = K(1 - \Omega) \left[\sum_{n=0}^{\infty} (2\Omega - (1 - K)\Omega^2)^n \right], \quad (3.15)$$

and evaluate the contribution of each term in the series to a typical state in the Fock space, for example $L_{-2}|0\rangle$. We have found numerically that the contributions to this coefficient eventually grow exponentially with n . By contrast, contributions from the analogous sum at $a = 1$ decay quite rapidly (as $1/n^4$) to give the coefficient $2.86 L_{-2}|0\rangle$.

3.1 Topology of Half-Brane Solutions

Though half-brane solutions are not real, every half-brane solution is related to its conjugate by a complex gauge transformation:

$$\Psi^\dagger = U^{-1}(Q + \Psi)U. \quad (3.16)$$

The required U is straightforward to compute (see appendix B of [19]) and is regular, in as far as the solutions themselves are regular. This raises an interesting issue. From the perspective of gauge invariant observables, a solution satisfying (3.16) is naively equivalent to

a real solution. In fact, such solutions have been useful for studying marginal deformations with singular OPEs[30, 31], solutions in Berkovits' string field theory[26, 32, 33, 34, 35], and the tachyon vacuum [19].

A second thought, however, reveals that (3.16) is not quite enough to guarantee the reality of observables. We must also require that U can be implemented as a sequence of infinitesimal gauge transformations, that is, U can be continuously deformed to the identity. Remarkably, for half-brane solutions, this appears not to be possible. That is, U is a topologically nontrivial gauge transformation. This is the first explicit example of a topologically nontrivial gauge transformation in string field theory, and is especially interesting since the topology is not related to any spacetime geometry in the $\alpha' \rightarrow 0$ limit, but appears to be intrinsic to the internal structure of the string.

To start we must define what it means to “continuously deform” the gauge transformation U . For simplicity, we will restrict ourselves to the K, B, c, G subalgebra, though we presume that our results are more general. We assume that a continuous deformation of the gauge transformation U will effect a continuous deformation of half brane solutions, in the following sense:

Definition 1. (*Continuity.*) Let $\Psi(t), t \in [0, 1]$ be a 1-parameter family of half-brane solutions in the K, B, c, G subalgebra. We say that this family is **continuous** only if $f_+(K, t)$ and $f_-(K, t)$, defined via (3.3), satisfy the following properties

A1)-A4) $f_+(K, t)$ and $f_-(K, t)$ satisfy conditions **1)-4)** for every fixed $t \in [0, 1]$.

B) $f_+(K, t)$ and $f_-(K, t)$ are continuous functions of K and t for $K \geq 0$ and $t \in [0, 1]$.

Conditions **A1)-A4)** ensure that $\Psi(t)$ is a regular half-brane solution for all t . Condition **B)** ensures that there are no “jumps” as we change t , that is, f_+ and f_- should change continuously with t if $\Psi(t)$ does. We now come to our central claim:

Theorem 2. *There is no continuous 1-parameter family of half-brane solutions $\Psi(t), t \in [0, 1]$ in the K, B, c, G subalgebra such that $\Psi(0) = \Psi(1)^\dagger$.*

This means, in particular, that the gauge transformation U relating a half-brane solution to its conjugate cannot be continuously deformed to the identity.

Proof. Since $\Psi(0) = \Psi(1)^\dagger$, the family of states $f_+(K, t)$ and $f_-(K, t)$ associated with $\Psi(t)$ must satisfy the boundary condition,

$$f_+(K, 0) = f_+(K, 1)^*, \quad f_-(K, 0) = f_-(K, 1)^*. \quad (3.17)$$

Analogous to the proof of Theorem 1, we will show that this boundary condition is incompatible with the continuity conditions stated above. In particular, we will show that the boundary condition, together with conditions **A1)**-**A3)** and **B)** imply that the field

$$D = (1 - f_+)^2 - K f_-^2 \quad (3.18)$$

has a zero at some point (K, t) . Therefore condition **A4)** is violated, and the sought after continuous family of solutions does not exist.

It is useful to think of K and t as coordinates on a semi-infinite strip

$$\Sigma = \mathbb{R}_+ \otimes [0, 1], \quad (3.19)$$

Consider the function

$$\Theta|_{\delta\Sigma} = \left. \frac{D}{|D|} \right|_{\delta\Sigma}, \quad (3.20)$$

which maps the boundary of Σ into complex numbers of unit modulus. The boundary includes the point at $K = \infty$, so that $\delta\Sigma$ has the topology of a circle. We make the following claims:

Claim 1. $\Theta|_{\delta\Sigma}$ is a continuous map from $\delta\Sigma$ into complex numbers of unit modulus.

Proof. By assumption we take the half-brane solution and its conjugate at $t = 0$ and $t = 1$ to be well-defined solutions. Therefore, D cannot have any zeros for positive K at $t = 0$ and $t = 1$. Conditions **A1)**, **A2)**, **B)** then imply continuity on all of $\delta\Sigma$. \square

Claim 2. If $\Theta|_{\delta\Sigma}$ has nonzero winding number, then D has a zero inside Σ .

Proof. Suppose D has no zeros in Σ . Since **B)** implies that D is continuous, we can extend $\Theta|_{\delta\Sigma}$ to a continuous function on the entire semi-infinite strip by simply taking $\Theta = \frac{D}{|D|}$. Shrinking $\delta\Sigma$ to a point, this function gives a continuous homotopy from $\Theta|_{\delta\Sigma}$ to the identity map. Since the identity map has zero winding number, the result follows. \square

Claim 3. The winding number of $\Theta|_{\delta\Sigma}$ is odd.

Proof. The proof of this claim is the most technical part of the argument. As a first step, it is helpful to introduce a notion of “winding number” for maps from a closed interval into complex numbers of unit modulus. Let g be a continuous map from a closed oriented interval I into complex numbers of unit modulus. We can lift g to a continuous map

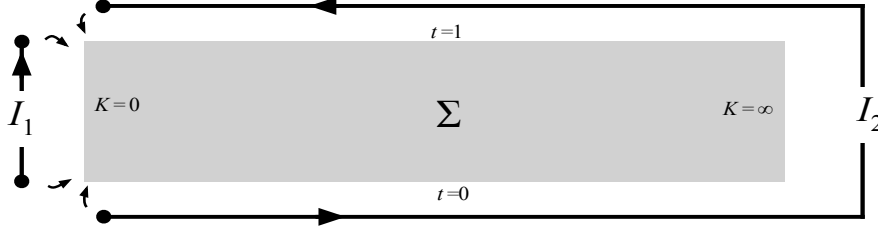


Figure 3.2: The boundary segments I_1 and I_2 of Σ .

$\phi : I \rightarrow \mathbb{R}$ such that $g = e^{i\phi}$. Parameterizing I by $\lambda \in [0, 1]$, we define the *winding number* of g to be the unique integer n such that

$$\phi(1) - \phi(0) = 2\pi n + R, \quad 0 \leq R < 2\pi. \quad (3.21)$$

We will write $n = w[g]$.

Now consider two closed oriented intervals I_1 and I_2 which intersect at their endpoints to form a circle S^1 . Assume that the orientation of I_1 is the same as that of the circle, and the orientation of I_2 is opposite. If g is a continuous map from S^1 into complex numbers of unit modulus, then

$$w[g] = w[g|_{I_1}] - w[g|_{I_2}], \quad (3.22)$$

where $g|_{I_1}$ and $g|_{I_2}$ is the restriction of g to the intervals I_1 and I_2 , respectively. The proof is straightforward.

We compute the winding number of $\Theta|_{\delta\Sigma}$ by splitting $\delta\Sigma$ into two segments, computing the winding numbers on each segment separately, and taking the difference following (3.22). The segments will be:

I_1 : the $K = 0$ boundary of Σ ,

I_2 : the $t = 0$ and $t = 1$ boundaries of Σ , connected through $K = \infty$.

See figure 3.2. First we compute the winding number of $\Theta|_{I_1}$ in terms of the winding number of the function

$$\frac{f_-}{|f_-|} \Big|_{I_1} = e^{i\theta}, \quad (3.23)$$

Conditions **A1**) and **B**) implies that θ is a continuous map from I_1 into \mathbb{R} . The winding number of (3.23) is the integer n satisfying

$$\theta(1) - \theta(0) = 2\pi n + R \quad 0 \leq R < 2\pi. \quad (3.24)$$

From **A1)** we also have

$$\Theta|_{I_1} = e^{i(2\theta+\pi)}. \quad (3.25)$$

The winding number of $\Theta|_{I_1}$ follows:

$$(2\theta(1) + \pi) - (2\theta(0) + \pi) = 4\pi n + 2R. \quad (3.26)$$

If $0 \leq R < \pi$, the winding number is $2n$, and if $\pi \leq R < 2\pi$ the winding number is $2n + 1$. Thus

$$\begin{aligned} w[\Theta|_{I_1}] &\in 2\mathbb{Z} && \text{if } 0 \leq R < \pi, \\ w[\Theta|_{I_1}] &\in 2\mathbb{Z} + 1 && \text{if } \pi \leq R < 2\pi. \end{aligned} \quad (3.27)$$

Now compute the winding number on I_2 . Parameterize I_2 by $\lambda \in [0, 1]$ such that: 1) $\lambda = 0$ corresponds to the corner of the strip where K and t both vanish; 2) $\lambda = \frac{1}{2}$ corresponds to the point at $K = \infty$, and 3) $\lambda = 1$ corresponds to the corner where K vanishes and $t = 1$. Also write

$$\Theta|_{I_2} = e^{i\psi}, \quad (3.28)$$

where ψ is a continuous map from I_2 into \mathbb{R} . Because of (3.17), $\Theta|_{I_2}$ evaluated on the $t = 1$ boundary is the complex conjugate of its value on the $t = 0$ boundary, which implies

$$\psi(1) - \psi(0) = 2(\psi(1) - \psi(\frac{1}{2})). \quad (3.29)$$

Continuity requires that $\Theta = 1$ at $K = \infty$, and therefore $\psi(\frac{1}{2}) = 2\pi m$ for some integer m :

$$\psi(1) - \psi(0) = 2\psi(1) - 4\pi m. \quad (3.30)$$

Comparing with equation (3.25) we are free to assume

$$\psi(1) = 2\theta(1) + \pi \quad (3.31)$$

Also (3.17) implies

$$\theta(1) = -\theta(0) + 2\pi k, \quad (3.32)$$

for some integer k . Therefore

$$\begin{aligned} \psi(1) - \psi(0) &= 4\theta(1) + 2\pi(-2m + 1) \\ &= 2(\theta(1) - \theta(0)) + 2\pi(2k - 2m + 1). \end{aligned} \quad (3.33)$$

Now we substitute equation (3.24) we find

$$\psi(1) - \psi(0) = 2\pi(2n + 2k - 2m + 1) + 2R. \quad (3.34)$$

Therefore

$$\begin{aligned} w[\Theta|_{I_2}] &\in 2\mathbb{Z} + 1 && \text{if } 0 \leq R < \pi, \\ w[\Theta|_{I_2}] &\in 2\mathbb{Z} && \text{if } \pi \leq R < 2\pi. \end{aligned} \quad (3.35)$$

Now we invoke (3.22) to find the final result:

$$w[\Theta|_{\delta\Sigma}] = w[\Theta|_{I_1}] - w[\Theta|_{I_2}] \in 2\mathbb{Z} + 1. \quad (3.36)$$

□

Since the winding number of $\Theta|_{\delta\Sigma}$ is odd, it cannot be zero. Therefore D must vanish at some point inside the semi-infinite strip. This completes the proof of Theorem 2. □

Theorem 2 implies that half-brane solutions come in at least two topologically distinct sectors in the K, B, c, G subalgebra. With some extra work, one can show that there are precisely two. It would be nice to find a way to characterize these sectors in terms of an invariant which is computable in terms of f_+ and f_- . A related question is that, though we have claimed that every half-brane solution is physically distinguishable from its conjugate on account of a topologically nontrivial gauge transformation, we have not found an observable which would actually distinguish between a half-brane solution and its conjugate in practice. The energy and closed string overlap, computed in section 5, are real observables for all half-brane solutions. Finding an observable which can detect the failure of the reality condition would give much insight into the physical significance of half-brane solutions, as well as their topological structure.

4 Regularization and Phantom Piece

In this section we discuss the regularization and phantom term for the half-brane solution. For clarity we focus on the Schnabl-like half-brane solution, though the discussion can be extended to solutions based on more general choices of f_{\pm} .

The Schnabl-like solution (2.27) can be written in the form

$$\Psi_{\text{Sch}} = \left[c \frac{KB}{1 - (1 + iaG)\Omega} c + B\gamma^2 \right] (1 + iaG)\Omega. \quad (4.1)$$

It is useful[15] to replace the factor between the c insertions by the partial sum of a geometric series, with the appropriate “error term”:

$$\frac{K}{1 - (1 + iaG)\Omega} = \sum_{n=0}^N K[(1 + iaG)\Omega]^n + \frac{K}{1 - (1 + iaG)\Omega} [(1 + iaG)\Omega]^{N+1}. \quad (4.2)$$

With this substitution, the solution can be written in the form

$$\Psi_{\text{Sch}} = \Psi_{N+1} - \sum_{n=0}^N \psi'_n + \Gamma, \quad (4.3)$$

where, reflecting the notation of Schnabl[21], we have defined the fields

$$\psi'_n = -cKB[(1 + iaG)\Omega]^n c[(1 + iaG)\Omega] \quad (4.4)$$

$$\Gamma = B\gamma^2[(1 + iaG)\Omega] \quad (4.5)$$

$$\Psi_{N+1} = c \left(\frac{KB}{1 - (1 + iaG)\Omega} [(1 + iaG)\Omega]^{N+1} \right) c[(1 + iaG)\Omega] \quad (4.6)$$

Since we have just made a trivial substitution, (4.3) is equal to the Schnabl-like solution for all N . However this expression is most useful in the $N \rightarrow \infty$ limit. In this limit Ψ_{N+1} becomes the so-called “phantom term,” and vanishes in the Fock space. Compared with phantom terms for the tachyon vacuum solution[15, 21, 36], the large N limit of Ψ_{N+1} is novel and requires careful treatment.

To understand the phantom term we should study the large N behavior of the string field

$$[(1 + iaG)\Omega]^N. \quad (4.7)$$

It is helpful to decompose this into $\text{GSO}(\pm)$ components as follows:

$$[(1 + iaG)\Omega]^N = X_N + (iaNG)Y_N, \quad (4.8)$$

where

$$\begin{aligned} X_N &= \frac{1}{2} [(1 + ia\sqrt{K})^N + (1 - ia\sqrt{K})^N] \Omega^N, \\ Y_N &= \frac{1}{2iaN\sqrt{K}} [(1 + ia\sqrt{K})^N - (1 - ia\sqrt{K})^N] \Omega^N. \end{aligned} \quad (4.9)$$

Above we introduced \sqrt{K} formally in order to write closed form expressions for the sums:

$$\begin{aligned} X_N &= \sum_{0 \leq k \leq N/2} \binom{N}{2k} a^{2k} (-K)^k \Omega^N, \\ Y_N &= \frac{1}{N} \sum_{0 \leq k \leq \frac{N-1}{2}} \binom{N}{2k+1} a^{2k} (-K)^k \Omega^N. \end{aligned} \quad (4.10)$$

A naive argument suggests that the large N limit of X_N and Y_N should be divergent. Note that while $(-K)^k \Omega^N$ vanishes as a power $\frac{1}{N^{2+k}}$ in the Fock space, the binomial coefficients grow very rapidly, so for fixed k and large N a term in the sum for X_N diverges as a power

$$\binom{N}{2k} (-K)^k \Omega^N \sim N^{k-2}, \quad (4.11)$$

Generically a sum of such terms would diverge faster than any power of N in the Fock space.

Miraculously, however, for a certain range of the parameter a X_N and Y_N converge to the sliver state:

$$\lim_{N \rightarrow \infty} X_N = \lim_{N \rightarrow \infty} Y_N = \Omega^\infty. \quad (4.12)$$

To see how this happens, consider the Fock space expansion for the wedge state Ω^α . We can write the expansion in the form

$$|\Omega^\alpha\rangle = \sum_{\vec{n}} P_{\vec{n}} \left(\frac{1}{\alpha+1} \right) L_{-n_q} \dots L_{-n_2} L_{-n_1} |0\rangle, \quad (4.13)$$

where $\vec{n} = (n_q, \dots, n_2, n_1)$ is a list of integers of arbitrary length satisfying

$$n_q \geq \dots \geq n_2 \geq n_1 \geq 2, \quad (4.14)$$

and $P_{\vec{n}}(x)$ are a collection of polynomials in x which determine the coefficients of $|0\rangle$ and its descendents. For example, up to level 4 the nonvanishing polynomials are

$$\begin{aligned} P_{()}(x) &= 1, & P_{(2)}(x) &= -\frac{1}{3} + \frac{4}{3}x^2, \\ P_{(2,2)}(x) &= \frac{1}{9} - \frac{8}{9}x^2 + \frac{16}{9}x^4, & P_{(4)}(x) &= \frac{1}{30} - \frac{16}{30}x^4. \end{aligned} \quad (4.15)$$

To compute X_N, Y_N , replace the factors of K multiplying Ω^N in (4.10) with derivatives via the formula,

$$(-K)^k \Omega^N = \left. \frac{d^k}{d\alpha^k} \Omega^\alpha \right|_{\alpha=N}, \quad (4.16)$$

and plug in the Fock space expansion (4.13). The coefficient of the state labeled by \vec{n} will then be

$$\begin{aligned} & \sum_{0 \leq k \leq N/2} \binom{N}{2k} a^{2k} \left. \frac{d^k}{d\alpha^k} P_{\vec{n}} \left(\frac{1}{\alpha+1} \right) \right|_{\alpha=N} \quad \text{for } X_N, \\ & \frac{1}{N} \sum_{0 \leq k \leq \frac{N-1}{2}} \binom{N}{2k+1} a^{2k} \left. \frac{d^k}{d\alpha^k} P_{\vec{n}} \left(\frac{1}{\alpha+1} \right) \right|_{\alpha=N} \quad \text{for } Y_N. \end{aligned} \quad (4.17)$$

The miracle of convergence as $N \rightarrow \infty$ is due to the following identities:

$$\lim_{N \rightarrow \infty} \left[\sum_{0 \leq k \leq N/2} \binom{N}{2k} a^{2k} \frac{d^k}{d\alpha^k} \frac{1}{(\alpha+1)^h} \Big|_{\alpha=N} \right] = 0, \quad (4.18)$$

$$\lim_{N \rightarrow \infty} \left[\sum_{0 \leq k \leq \frac{N-1}{2}} \binom{N}{2k+1} a^{2k} \frac{d^k}{d\alpha^k} \frac{1}{(\alpha+1)^h} \Big|_{\alpha=N} \right] = 0, \quad (4.19)$$

which assume

$$h > 0, \quad a \in [-\sqrt{2}, \sqrt{2}]. \quad (4.20)$$

Thus taking the $N \rightarrow \infty$ of equation (4.17), all nonzero powers of $\frac{1}{1+\alpha}$ are killed, leaving

$$\begin{aligned} P_{\vec{n}}(0) & \quad \text{for} \quad \lim_{N \rightarrow \infty} X_N, \\ P_{\vec{n}}(0) & \quad \text{for} \quad \lim_{N \rightarrow \infty} Y_N. \end{aligned} \quad (4.21)$$

These are exactly the coefficients of the sliver state.

To prove the identities (4.18) and (4.19), it is helpful to represent the ratio $\frac{1}{1+\alpha}$ as an integral:

$$\frac{1}{(\alpha+1)^h} = \frac{1}{(h-1)!} \int_0^\infty dt t^{h-1} e^{-(\alpha+1)t}. \quad (4.22)$$

Substituting in (4.18) converts the sum into a simple integral:

$$\begin{aligned} & \sum_{0 \leq k \leq N/2} \binom{N}{2k} a^{2k} \frac{d^k}{d\alpha^k} \frac{1}{(\alpha+1)^h} \Big|_{\alpha=N} \\ &= \frac{1}{(h-1)!} \int_0^\infty dt t^{h-1} e^{-t} \frac{1}{2} \left(\left[(1 + ia\sqrt{t})e^{-t} \right]^N + \left[(1 - ia\sqrt{t})e^{-t} \right]^N \right). \end{aligned} \quad (4.23)$$

Note the similarity of the integrand with (4.9). Let us assume that the two terms in the integrand should be bounded in absolute value in the $N \rightarrow \infty$ limit. This requires

$$|(1 \pm ia\sqrt{t})e^{-t}|^2 \leq 1, \quad (4.24)$$

which implies that a must be a real number in the range $[\sqrt{2}, -\sqrt{2}]$.¹¹ Now we can compute the $N \rightarrow \infty$ limit in the integrand by simply noting

$$\lim_{N \rightarrow \infty} \left[(1 \pm ia\sqrt{t})e^{-t} \right]^N = \begin{cases} 1 & \text{at } t = 0 \\ 0 & \text{otherwise} \end{cases}. \quad (4.25)$$

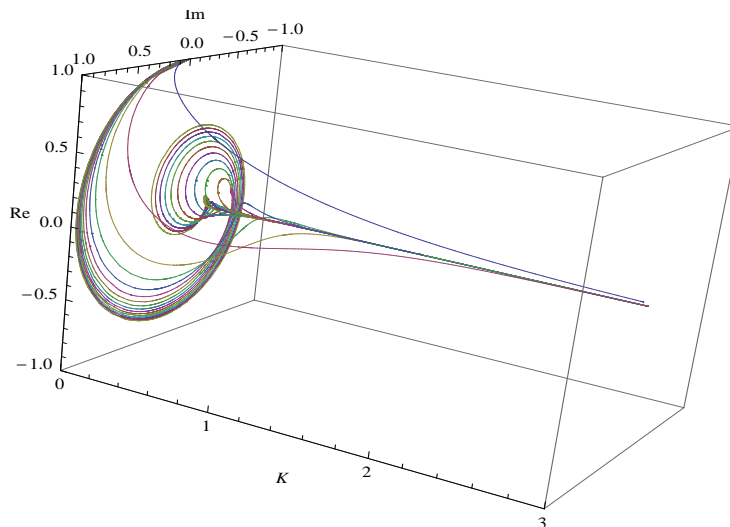


Figure 4.1: Plots of $[(1 + i\sqrt{K})e^{-K}]^N$ as a function of K for $N = 1, 3, 5, \dots, 31$. The vertical axis is the real part, the forward axis is the imaginary part, and the horizontal axis is K .

Since the integrand vanishes almost everywhere, the sum (4.18) must vanish. This completes the proof that X_N and Y_N approach this sliver state in the large N limit.

Perhaps this result is not surprising, since the sliver state is the only projector which could have emerged from the $N \rightarrow \infty$ limit. What is more interesting is the manner in which X_N and Y_N approach the sliver. Recall that the sliver state corresponds to a function of K which takes the value 1 at $K = 0$ and vanishes everywhere else. X_N and Y_N approach the sliver by a sequence of functions

$$[(1 + ia\sqrt{K})e^{-K}]^N. \quad (4.26)$$

A plot of the real and imaginary parts of these functions is shown in figure 4.1. The plot reveals a spiral, which as N increases, winds increasingly many times around the K axis and becomes increasingly damped away from $K = 0$. As $N \rightarrow \infty$, the function $[(1 + ia\sqrt{K})e^{-K}]^N$ vanishes for $K > 0$, but for infinitesimal K winds so many times that it fills the whole unit disk in the complex plane. Now compare this to how wedge states approach the sliver at large wedge angle, corresponding to the sequence of functions e^{-NK} . The large N limit of e^{-NK} reveals none of the highly oscillatory behavior seen in figure 4.1. We can formalize this qualitative observation as follows. Assume, following a

¹¹Numerical computations suggest that that (4.18) may hold even in a limited region of the complex plane around the line segment $-\sqrt{2} \leq a \leq \sqrt{2}$. We do not have an analytic proof, however.

proposal of Rastelli[28], that convergence in the wedge algebra is determined by the norm

$$||f(K)|| = \sup |f(K)|. \quad (4.27)$$

In the large N limit, X_N and Y_N could be considered “close” to a wedge state if there were a set of numbers $\sigma(N), \rho(N)$ such that the sequence of norms

$$||X_N - \Omega^{\sigma(N)}||, \quad ||Y_N - \Omega^{\rho(N)}|| \quad (4.28)$$

converged to zero. However, this is impossible; $\Omega^{\sigma(N)}$ only takes values between 0 and 1, whereas X_N and Y_N take all values in the interval $[-1, 1]$ for sufficiently large N . We can make a similar observation by looking at the Fock space expansion. In appendix D we compute the leading order corrections to the sums (4.18) and (4.19):

$$\begin{aligned} \sum_{0 \leq k \leq N/2} \binom{N}{2k} a^{2k} \frac{d^k}{d\alpha^k} \frac{1}{(\alpha+1)^h} \Big|_{\alpha=N} &= (-1)^h \frac{2(2h-1)!}{(h-1)!} \frac{1}{(aN)^{2h}} + \dots, \\ \sum_{0 \leq k \leq \frac{N-1}{2}} \binom{N}{2k+1} a^{2k} \frac{d^k}{d\alpha^k} \frac{1}{(\alpha+1)^h} \Big|_{\alpha=N} &= (-1)^{h+1} \frac{4N(2h-3)!}{(h-2)!} \frac{1}{(aN)^{2h}} + \dots \end{aligned} \quad (4.29)$$

Plugging these into (4.17), we find that the leading order correction to X_N and Y_N for large N is numerically quite different from that of a wedge state with large wedge angle, especially due to the h -dependent factors in (4.29). In principle these differences could have an important effect on calculations involving the phantom term. Therefore, though X_N and Y_N approach the sliver, it is not the “same” sliver as Ω^N for large N .¹²

This motivates us to introduce a new class of states, more general than wedge states, which could describe the large N behavior of X_N and Y_N :

$$e^{-\alpha K} e^{i\beta G} = \Omega^\alpha \left[\cos(\beta\sqrt{K}) + iG \frac{\sin(\beta\sqrt{K})}{\sqrt{K}} \right]. \quad (4.30)$$

We will call these *super-wedge states*. It turns out that super-wedge states cannot be easily described as a superposition of wedge surfaces. We will say more about how these states can be constructed in the next section and appendix D. Consider the large N limit of the super-wedge state

$$\left(e^{iaG} \Omega^{1-\frac{a^2}{2}} \right)^N = \hat{X}_N + (iaNG) \hat{Y}_N. \quad (4.31)$$

¹²Note that the analog of X_N, Y_N for the tachyon vacuum solutions of [36] is the infinite power $f(K)^N$, where f is some function of K satisfying the constraints $f(0) = 1, f'(0) = -\gamma < 0$ and $|f(K)| \leq 1$. We can show that the leading correction to the Fock space coefficients for any such state match those of a wedge state $\Omega^{\gamma N}$. Moreover, the sequence of norms $||f^N - \Omega^{\gamma N}||$ converges to zero. In this sense, $f(K)^N$ is approximately equal to the wedge state $\Omega^{\gamma N}$ for large N .

One can check that the sequence of norms

$$||X_N - \hat{X}_N||, \quad ||Y_N - \hat{Y}_N|| \quad (4.32)$$

converges to zero, and moreover the leading large N correction to the Fock space coefficients of (\hat{X}_N, \hat{Y}_N) match those of (X_N, Y_N) . We can therefore simplify the phantom piece for large N by replacing

$$\left[(1 + iaG)\Omega\right]^N \rightarrow \left(e^{iaG}\Omega^{1-\frac{a^2}{2}}\right)^N, \quad (4.33)$$

and

$$\frac{K}{1 - (1 + iaG)\Omega} \rightarrow \frac{i}{a}G, \quad (4.34)$$

which is the leading term in the \mathcal{L}^- level expansion of this factor.¹³ The phantom term therefore simplifies to

$$\hat{\Psi}_N = \frac{i}{a}c \left[GB \left(e^{iaG}\Omega^{1-\frac{a^2}{2}} \right)^N \right] c (1 + iaG)\Omega, \quad (4.35)$$

and we can express the Schnabl-like solution in regularized form,

$$\Psi_{\text{Sch}} = \lim_{N \rightarrow \infty} \left[\hat{\Psi}_N - \sum_{n=0}^N \psi'_n + \Gamma \right]. \quad (4.36)$$

Unlike (4.3), this expression is only valid in the $N \rightarrow \infty$ limit.

A few comments about the parameter a . Note that the phantom term (4.35) is manifestly singular as a approaches zero. This corresponds to the fact that at $a = 0$ Ψ_{Sch} is actually a solution for the tachyon vacuum, which requires a different phantom piece. Also note we needed to fix a to lie within a restricted range $-\sqrt{2} \leq a \leq \sqrt{2}$. We do not know whether this bound reflects a limitation of the regularization (4.36) or a deeper problem with the solution when a sits outside the restricted range.

4.1 An Aside: General States in the Wedge Algebra

Up to now, most states in the wedge algebra have been assumed to take the form

$$f(K) = \int_0^\infty dt \tilde{f}(t) \Omega^t, \quad (4.37)$$

¹³Sometimes it is necessary to include more than the leading term in the \mathcal{L}^- level expansion[15], but ignore this possibility for simplicity.

and thus are a “linear combination” of wedge states. When this integral converges, the function $f(K)$ can be identified with the Laplace transform of $\tilde{f}(t)$. However, the set of functions which can be represented as a Laplace transform in this sense is limited. Here we would like to give a more general construction of $f(K)$ motivated by our analysis of the phantom term.

Consider the space of polynomials over a variable x , which we denote $\mathbb{R}[x]$. Here, x will be identified with the ratio $\frac{1}{1+\alpha}$ in the Fock space expansion of the wedge state Ω^α . Given a suitable function $f(t)$, we define a linear functional L_f on $\mathbb{R}[x]$ as follows:

$$\begin{aligned} L_f(x^0) &= f(0), \\ L_f(x^1) &= \int_0^\infty dt f(t) e^{-t}, \\ &\vdots \\ L_f(x^h) &= \frac{1}{(h-1)!} \int_0^\infty dt t^{h-1} f(t) e^{-t}. \end{aligned} \tag{4.38}$$

With the help of this functional, we define the state,

$$f(K) = L_f(\Omega^\alpha), \quad x = \frac{1}{1+\alpha}. \tag{4.39}$$

This should be understood as a definition of $f(K)$ in the Fock space. One can check that (4.39) and (4.37) give exactly the same expressions for the coefficients of $f(K)$ in the domain where both formulas are defined. However (4.39) is much more general. For example, (4.39) allows one to construct a string field for any $f(K)$ in the algebra of bounded, continuous functions on the positive real line. The existence of such string fields is implied by Rastelli’s proposal for the definition of the algebra of wedge states[28].

The simplest example of a state which we can construct from (4.39), but not (4.37), is a wedge state with complex wedge angle

$$\Omega^{\alpha+i\beta}. \tag{4.40}$$

The linear functional (4.38) is

$$L_{\Omega^{\alpha+i\beta}}(x^h) = \frac{1}{(h-1)!} \int_0^\infty dt t^{h-1} e^{-(\alpha+1)t+i\beta t} = \frac{1}{(1+\alpha+i\beta)^h}, \quad \alpha > -1. \tag{4.41}$$

This is the obvious analytic continuation of the usual Fock space expansion of a wedge state to complex wedge angle. But note that the linear functional only converges if

$\alpha > -1$. This “explains” the curious relation between the Fock space coefficients of wedge states with positive and negative wedge angle:

$$\Omega^\alpha = \Omega^{-2-\alpha} \quad ? \quad (4.42)$$

In reality an inverse wedge state for $\alpha < -1$ is divergent in the Fock space, and not the analytic continuation of (4.41).

Another example is the state

$$f(K) = \frac{\lambda K \Omega}{1 - \lambda \Omega}. \quad (4.43)$$

This is the pure gauge solution of Schnabl[21], either in the ghost number zero toy model or in the ghost number one case after ignoring the B, c insertions. For $|\lambda| < 1$ we can express this as a Laplace transform by making a geometric series expansion of the denominator, but for $|\lambda| > 1$ this series is divergent. Still we can define the linear functional

$$L_f(x^h) = \frac{1}{(h-1)!} \int_0^\infty dt \frac{\lambda t^h e^{-2t}}{1 - \lambda e^{-t}} = \lambda h \Phi(\lambda, h, 2), \quad \lambda \not\approx 1, \quad (4.44)$$

where $\Phi(z, s, v)$ is the Lerch function. The integral is absolutely convergent as long as λ is not a real number greater than one. This suggests that the pure gauge solutions are defined even for $\lambda < -1$, though the geometric series is divergent. This would bring Schnabl’s pure gauge solutions into line with the pure gauge solutions of [19], which are also nonsingular for $\lambda < -1$.

As a final (somewhat peculiar) example, consider the characteristic function on the interval $[a, b] > 0$:

$$\mathbf{1}_{[a,b]}(K) = \begin{cases} 1 & \text{for } a \leq K \leq b \\ 0 & \text{otherwise} \end{cases}. \quad (4.45)$$

Formally these are all projectors in the wedge algebra, and if the interval does not include 0, the projectors are orthogonal to the sliver state. There is no hope of representing such states as a Laplace transform (4.37), but we can still define them using the functional

$$L_{\mathbf{1}_{[a,b]}}(x^h) = \sum_{n=0}^{h-1} \frac{a^n}{n!} e^{-a} - \sum_{n=0}^{h-1} \frac{b^n}{n!} e^{-b}. \quad (4.46)$$

Note that these projectors are infinite rank. Unlike the sliver, they are difficult to reach by taking the infinite power of a “reasonable” $f(K)$ in the wedge algebra. While one

can apparently define such an $f(K)$ using (4.39), it would have to satisfy the awkward constraint of being exactly equal to unity on the interval $[a, b]$, and strictly less than unity in absolute value outside that interval.

We have not confirmed whether any of these states behave as expected under star multiplication. Since they are not linear combinations of surfaces, one cannot study their star products using the usual gluing rules of conformal field theory. Nevertheless, these states are concrete constructions which could be interesting for future study.

5 Observables

5.1 Gauge Invariant Overlap for Simple Half-Brane Solution

We start with the simplest computation, that of the gauge invariant overlap for the simple half-brane solution (2.25):

$$W(\Psi_{\text{simp}}, \mathcal{V}) = \langle\langle \Psi_{\text{simp}} \rangle\rangle_{\mathcal{V}}. \quad (5.1)$$

Here the bracket $\langle\langle \cdot \rangle\rangle_{\mathcal{V}}$ is defined in the same way as the vertex $\langle\langle \cdot \rangle\rangle$ (see (A.1)) except the picture changing operator Y_{-2} is replaced by an on shell closed string vertex operator $\mathcal{V}(i)$ inserted at the midpoint. We assume \mathcal{V} is an NS-NS closed string vertex operator of the form,

$$\mathcal{V}(z) = c\tilde{c}e^{-\phi}e^{-\tilde{\phi}}\mathcal{O}^{(\frac{1}{2}, \frac{1}{2})}(z, \bar{z}), \quad (5.2)$$

where $\mathcal{O}^{(\frac{1}{2}, \frac{1}{2})}$ is a weight $(\frac{1}{2}, \frac{1}{2})$ superconformal matter primary. We work in the small Hilbert space, so the ξ zero mode is absent. If the interpretation of Ellwood[37] is correct, the gauge invariant overlap should represent the shift in the closed string tadpole of the solution relative to the perturbative vacuum.

Plugging in the simple solution (2.25) into the overlap, the BRST exact term does not contribute since \mathcal{V} is on-shell. Furthermore the $\text{GSO}(-)$ component vanishes in the correlator. This leaves

$$W(\Psi_{\text{simp}}, \mathcal{V}) = - \left\langle\left\langle cGBc \frac{G}{1+K} \right\rangle\right\rangle_{\mathcal{V}}. \quad (5.3)$$

Now expand $\frac{1}{1+K}$ in terms of wedge states

$$- \left\langle\left\langle cGBc \frac{G}{1+K} \right\rangle\right\rangle_{\mathcal{V}} = - \int_0^\infty dt e^{-t} \left\langle\left\langle cGBcG\Omega^t \right\rangle\right\rangle_{\mathcal{V}}. \quad (5.4)$$

Note

$$cGBcG\Omega^t = t^{\frac{1}{2}\mathcal{L}^-}(cGBcG\Omega). \quad (5.5)$$

Since the operator $t^{\frac{1}{2}\mathcal{L}^-}$ is a reparameterization, it leaves the bracket invariant. We can then easily evaluate the integral over t to find

$$W(\Psi_{\text{simp}}, \mathcal{V}) = -\langle\langle cGBcG\Omega \rangle\rangle_{\mathcal{V}}. \quad (5.6)$$

Now commute the leftmost G insertion towards the other G insertion:

$$\begin{aligned} W(\Psi_{\text{simp}}, \mathcal{V}) &= -\langle\langle cB(cG + \delta c)G\Omega \rangle\rangle_{\mathcal{V}} \\ &= -\langle\langle cK\Omega \rangle\rangle_{\mathcal{V}} + 2i\langle\langle c\gamma BG\Omega \rangle\rangle_{\mathcal{V}}. \end{aligned} \quad (5.7)$$

To compute the first term note

$$-cK\Omega = \frac{1}{2}\mathcal{L}^-(c\Omega) + c\Omega. \quad (5.8)$$

Since \mathcal{L}^- kills the bracket, this leaves

$$W(\Psi_{\text{simp}}, \mathcal{V}) = \langle\langle c\Omega \rangle\rangle_{\mathcal{V}} + 2i\langle\langle c\gamma BG\Omega \rangle\rangle_{\mathcal{V}}. \quad (5.9)$$

Now focus on the second term. Using cyclicity we can rewrite it in the form,

$$2i\langle\langle c\gamma BG\Omega \rangle\rangle_{\mathcal{V}} = i\left\langle\left\langle G(c\gamma B\Omega) + (c\gamma B\Omega)G \right\rangle\right\rangle_{\mathcal{V}}. \quad (5.10)$$

Since γ carries odd worldsheet spinor number, the G anticommutator above is exactly the worldsheet supersymmetry variation δ :

$$\delta(c\gamma B\Omega) = G(c\gamma B\Omega) + (c\gamma B\Omega)G. \quad (5.11)$$

Therefore we can explicitly eliminate G :

$$\begin{aligned} 2i\langle\langle c\gamma BG\Omega \rangle\rangle_{\mathcal{V}} &= i\langle\langle \delta(c\gamma B\Omega) \rangle\rangle_{\mathcal{V}} \\ &= \left\langle\left\langle -2\gamma^2 B\Omega + \frac{1}{2}c\partial cB\Omega \right\rangle\right\rangle_{\mathcal{V}}, \end{aligned} \quad (5.12)$$

where we used the derivation property of δ and the explicit variations (2.12). To eliminate the remaining B insertion we use the derivation \mathcal{B}^- , which satisfies

$$\frac{1}{2}\mathcal{B}^- K = B, \quad \frac{1}{2}\mathcal{B}^- \{B, c, \text{ or } \gamma\} = 0, \quad (5.13)$$

and leaves the vertex invariant. This allows us to express

$$-2\gamma^2 B\Omega + \frac{1}{2}c\partial cB\Omega = \mathcal{B}^- \left[-2\gamma^2 \Omega - \frac{1}{2}cKc\Omega \right] - \frac{1}{2}c\Omega. \quad (5.14)$$

The \mathcal{B}^- term kills the bracket leaving

$$2i\langle\langle c\gamma BG\Omega\rangle\rangle_{\mathcal{V}} = -\frac{1}{2}\langle\langle c\Omega\rangle\rangle_{\mathcal{V}}. \quad (5.15)$$

Plugging into (5.9) and gives the final answer:

$$W(\Psi_{\text{simp}}, \mathcal{V}) = \frac{1}{2}\langle\langle c\Omega\rangle\rangle_{\mathcal{V}}. \quad (5.16)$$

This is exactly one-half the value of the overlap at the tachyon vacuum[37]. This confirms that the half-brane solutions are not gauge equivalent to either the tachyon vacuum or the perturbative vacuum (where the overlap vanishes identically). It also indicates that half-brane solutions must source closed strings with half the strength of a non-BPS D-brane.

5.2 Gauge Invariant Overlap for Schnabl-Like Half-Brane Solution

As a check on the consistency of our results, we would like to compute the gauge invariant overlap for the Schnabl-like solution. Plugging in (4.3) we find

$$\begin{aligned} W(\Psi_{\text{Sch}}, \mathcal{V}) &= \langle\langle \Psi_{\text{Sch}} \rangle\rangle_{\mathcal{V}}, \\ &= \langle\langle \Psi_{N+1} \rangle\rangle_{\mathcal{V}} - \sum_{n=0}^N \langle\langle \psi'_n \rangle\rangle_{\mathcal{V}} + \langle\langle \Gamma \rangle\rangle_{\mathcal{V}}. \end{aligned} \quad (5.17)$$

The second two terms do not contribute to the overlap. The easiest way to see this is to note that the overlap for the pure gauge solution

$$\Psi_{\lambda} = -\sum_{n=0}^{\infty} \lambda^n \psi'_n + \lambda \Gamma \quad (5.18)$$

must vanish order by order in λ by gauge invariance. Therefore (5.17) simplifies to

$$W(\Psi_{\text{Sch}}, \mathcal{V}) = \langle\langle \Psi_N \rangle\rangle. \quad (5.19)$$

This equation holds for any N , but we will be interested in the limit $N \rightarrow \infty$.

Plugging in the (4.6) for Ψ_N the overlap reduces to a sum of four terms:

$$\begin{aligned}
W(\Psi_{\text{Sch}}, \mathcal{V}) = & \left\langle\left\langle c \frac{KB(1-\Omega)}{(1-\Omega)^2 + a^2 K \Omega^2} X_N c \Omega \right\rangle\right\rangle_{\mathcal{V}} \\
& - Na^2 \left\langle\left\langle c \frac{K^2 B \Omega}{(1-\Omega)^2 + a^2 K \Omega^2} Y_N c \Omega \right\rangle\right\rangle_{\mathcal{V}} \\
& - a^2 \left\langle\left\langle c \frac{KGB\Omega}{(1-\Omega)^2 + a^2 K \Omega^2} X_N c G \Omega \right\rangle\right\rangle_{\mathcal{V}} \\
& - Na^2 \left\langle\left\langle c \frac{KGB(1-\Omega)}{(1-\Omega)^2 + a^2 K \Omega^2} Y_N c G \Omega \right\rangle\right\rangle_{\mathcal{V}}. \tag{5.20}
\end{aligned}$$

Now in each of the four terms expand all of the wedge state factors besides X_N and Y_N in powers of K :

$$\left\langle\left\langle c \frac{KB(1-\Omega)}{(1-\Omega)^2 + a^2 K \Omega^2} X_N c \Omega \right\rangle\right\rangle_{\mathcal{V}} = \sum_{m \geq 1, n \geq 1} C_{mn}^{(1)} \langle\langle c B K^m X_N c K^n \rangle\rangle_{\mathcal{V}}, \tag{5.21}$$

$$-Na^2 \left\langle\left\langle c \frac{K^2 B \Omega}{(1-\Omega)^2 + a^2 K \Omega^2} Y_N c \Omega \right\rangle\right\rangle_{\mathcal{V}} = \sum_{m \geq 1, n \geq 1} C_{mn}^{(2)} \langle\langle c B K^m Y_N c K^n \rangle\rangle_{\mathcal{V}}, \tag{5.22}$$

$$-a^2 \left\langle\left\langle c \frac{KGB\Omega}{(1-\Omega)^2 + a^2 K \Omega^2} X_N c G \Omega \right\rangle\right\rangle_{\mathcal{V}} = \sum_{m \geq 0, n \geq 0} C_{mn}^{(3)} \langle\langle c B G K^m X_N c G K^n \rangle\rangle_{\mathcal{V}}, \tag{5.23}$$

$$-Na^2 \left\langle\left\langle c \frac{KGB(1-\Omega)}{(1-\Omega)^2 + a^2 K \Omega^2} Y_N c G \Omega \right\rangle\right\rangle_{\mathcal{V}} = \sum_{m \geq 1, n \geq 0} C_{mn}^{(4)} \langle\langle c B G K^m Y_N c G K^n \rangle\rangle_{\mathcal{V}}, \tag{5.24}$$

where $C_{m,n}^{(a)}$ are constants. Let us focus on (5.23). To calculate the double sum, we should compute the traces

$$\langle\langle c B G K^m X_N c G K^n \rangle\rangle_{\mathcal{V}}. \tag{5.25}$$

Plugging in (4.10) for X_N this becomes

$$\langle\langle c B G K^m X_N c G K^n \rangle\rangle_{\mathcal{V}} = \sum_{0 \leq K \leq N/2} \binom{N}{2k} a^{2k} \frac{d^k}{d\alpha^k} \langle\langle c B G K^m \Omega^\alpha c G K^n \rangle\rangle_{\mathcal{V}} \Big|_{\alpha=N}. \tag{5.26}$$

Now reparameterize the bracket with \mathcal{L}^- to factor the α dependence:

$$\langle\langle c B G K^m X_N c G K^n \rangle\rangle_{\mathcal{V}} = \langle\langle c B G K^m \Omega c G K^n \rangle\rangle_{\mathcal{V}} \sum_{0 \leq K \leq N/2} \binom{N}{2k} a^{2k} \frac{d^k}{d\alpha^k} \frac{1}{\alpha^{m+n}} \Big|_{\alpha=N}. \tag{5.27}$$

We recognize the sum on the right hand side from the identity (4.18). Provided $a \in [-\sqrt{2}, \sqrt{2}]$ and $m + n > 0$, this vanishes in the $N \rightarrow \infty$ limit. Therefore if we take $N \rightarrow \infty$ only the $m = n = 0$ term in (5.23) contributes to the overlap:

$$-a^2 \lim_{N \rightarrow \infty} \left\langle\left\langle c \frac{KGB\Omega}{(1-\Omega)^2 + a^2 K\Omega^2} X_N c G \Omega \right\rangle\right\rangle_{\mathcal{V}} = -\langle\langle cBG\Omega cG \rangle\rangle_{\mathcal{V}}. \quad (5.28)$$

Now repeat this argument for equations (5.21), (5.22) and (5.24). However, this time the range of summation over m, n excludes all traces which could make a nonzero contribution in the $N \rightarrow \infty$ limit. So, in fact, (5.28) is the only contribution to the overlap in the $N \rightarrow \infty$ limit, and we find

$$W(\Psi_{\text{Sch}}, \mathcal{V}) = -\langle\langle cBG\Omega cG \rangle\rangle_{\mathcal{V}}. \quad (5.29)$$

With a few manipulations this can be rewritten,

$$W(\Psi_{\text{Sch}}, \mathcal{V}) = -2i\langle\langle c\gamma BG\Omega \rangle\rangle_{\mathcal{V}}. \quad (5.30)$$

From here on the derivation follows the steps given in (5.10)-(5.15), with an extra minus sign, to yield

$$W(\Psi_{\text{Sch}}, \mathcal{V}) = \frac{1}{2}\langle\langle c\Omega \rangle\rangle_{\mathcal{V}}. \quad (5.31)$$

in agreement with the overlap for the simple half-brane solution, (5.16).

5.3 Energy

Let us now calculate the energy. The energy can be computed from the on-shell action:

$$E = -S[\Psi] = -\frac{1}{6}\langle\langle \Psi Q \Psi \rangle\rangle. \quad (5.32)$$

We have only attempted this calculation for the simple half-brane solution (2.25), though the final answer should be the same for any sufficiently regular half-brane solution. Plugging in Ψ_{simp} we find the expression

$$E = -\frac{1}{6} \left[-\left\langle\left\langle cG B c \frac{1}{1+K} Q(cG B c) \frac{1}{1+K} \right\rangle\right\rangle + \left\langle\left\langle cG B c \frac{G}{1+K} Q(cG B c) \frac{G}{1+K} \right\rangle\right\rangle \right]. \quad (5.33)$$

To simplify we eliminate the G insertions by repeated use of the identity

$$\langle\langle G \Phi \rangle\rangle = \frac{1}{2}\langle\langle \delta \Phi \rangle\rangle. \quad (5.34)$$

The calculation is straightforward, but tedious; the repeated supersymmetry variations generate dozens of terms. We give some details in appendix E. In the end, all of the inner products can be evaluated with the correlation function

$$\left\langle Y_{-2} c(x_1) c(x_2) \gamma(y_1) \gamma(y_2) \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} b(z) \right\rangle_{C_L} = -\frac{L}{2\pi^2} (x_1 - x_2) \cos \frac{\pi(y_1 - y_2)}{L} \quad (5.35)$$

evaluated on a cylinder of circumference L . Adding the terms up, the energy turns out to be

$$E = -\frac{1}{4\pi^2}, \quad (5.36)$$

which is precisely $-1/2$ times the tension of the D-brane. Remarkably, this is consistent with the computation of the overlap.

6 Concluding Remarks

In this paper we have presented a new class of nonperturbative analytic solutions of cubic superstring field theory on a non-BPS D-brane. The nature of these solutions is fundamentally mysterious; they violate the reality condition, and appear not to exist in Berkovits string field theory. Probably they are only formal artifacts of the cubic equations of motion. However, their existence is nontrivial and seems significant. We hope that further study will shed light into what these solutions represent and why they exist.

One immediate consequence of our analysis is that the cubic and Berkovits equations of motion are not equivalent. The fact that half-brane solutions appear to come in two topologically distinct varieties, and the existence of a “tachyon vacuum” on a BPS D-brane, suggest that the failure of this equivalence is related to topological charge. A microscopic understanding of D-brane charges is one of the longstanding goals of string field theory. We hope that continued development along these lines will give further insight.

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A Vertices, Reality and Twist conjugation

In this appendix we discuss some important signs connected with the vertices and the reality condition in the $\text{GSO}(-)$ sector. Our discussion extends the classic analysis of Ohmori[6] to our preferred “left handed” star product convention[19].

Definition of vertices: In the left handed convention, the N -string vertex of open string fields $\Phi_k = \sigma_{i_k} \phi_k(0)|0\rangle$ is defined as a correlator on the upper half plane as follows:

$$\langle\langle \Phi_1, \Phi_2, \dots, \Phi_N \rangle\rangle = \frac{1}{2} \text{tr}(\sigma_3 \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_N}) \left\langle Y_{-2} f_{1,N} \circ \phi_1(0) f_{2,N} \circ \phi_2(0) \dots f_{N,N} \circ \phi_N(0) \right\rangle, \quad (\text{A.1})$$

where

$$Y_{-2} = Y(i) \tilde{Y}(i), \quad Y(z) = -\partial \xi e^{-2\phi} c(z), \quad (\text{A.2})$$

and the conformal maps defining the vertex are,

$$f_{k,N}(z) = -\cot \left[\frac{2}{N} \tan^{-1} z - \frac{\pi}{N} \left(k - \frac{1}{2} \right) \right]. \quad (\text{A.3})$$

Let us define

$$x_{k,N} = \cot \frac{\pi}{N} \left(k - \frac{1}{2} \right), \quad y_{k,N} = -\sqrt{\frac{2}{N}} \csc \frac{\pi}{N} \left(k - \frac{1}{2} \right). \quad (\text{A.4})$$

Then $f_{k,N}$ acts on a primary of weight h explicitly as

$$f_{k,N} \circ \phi(0) = (y_{k,N})^{(2h)} \phi(x_{k,N}), \quad (\text{A.5})$$

where the parentheses around $2h$ implies that h must be multiplied by two *before* the power of $y_{k,N}$ is taken. Note that

$$x_{1,N} > x_{2,N} > \dots > x_{N,N}, \quad (\text{A.6})$$

so the position of the vertex operator on the real axis *decreases* as we increase the string label k in the vertex. This is the hallmark of the left handed star product convention.

By contrast, in the right handed convention the N -string vertex would be defined as

$$\langle\langle \Phi_1, \Phi_2, \dots, \Phi_N \rangle\rangle_R = \frac{1}{2} \text{tr}(\sigma_3 \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_N}) \left\langle Y_{-2} \tilde{f}_{1,N} \circ \phi_1(0) \tilde{f}_{2,N} \circ \phi_2(0) \dots \tilde{f}_{N,N} \circ \phi_N(0) \right\rangle, \quad (\text{A.7})$$

where the superscript R reminds us that the vertex is defined in the right handed convention. The conformal maps $\tilde{f}_{k,N}$ are related to $f_{k,N}$ simply as,

$$\tilde{f}_{k,N}(z) = f_{N+1-k,N}(z). \quad (\text{A.8})$$

The positions of the vertex operators on the real axis are

$$\tilde{x}_{k,N} = x_{N+1-k,N}, \quad (\text{A.9})$$

so the position *increases* as we increase the string label. This is the hallmark of the right handed star product convention.

The vertices (A.1) and (A.7) implicitly define the open string star product. The left handed star product $\Psi\Phi$ and right handed star products $[\Psi\Phi]_R$ are related by the equation,

$$[\Psi\Phi]_R = (-1)^{E(\Psi)E(\Phi)+F(\Psi)F(\Phi)}\Phi\Psi. \quad (\text{A.10})$$

The sign appears from anticommuting vertex operators and internal CP factors.

Let us consider the 2-string vertex. Following [6], we define the action of the BPZ conformal map $I(z) = -\frac{1}{z}$ on a primary of weight h

$$I \circ \phi(z) = \phi(z)^\star = \frac{1}{z^{(2h)}}\phi\left(-\frac{1}{z}\right). \quad (\text{A.11})$$

By an $SL(2, \mathbb{R})$ transformation, one can then show that the 2-string vertex in the left and right handed conventions is given by,

$$\begin{aligned} \langle\langle \Phi_1, \Phi_2 \rangle\rangle &= \frac{1}{2}\text{tr}(\sigma_3\sigma_{i_1}\sigma_{i_2})\left\langle Y_{-2} \phi_1(0) \phi_2(0)^\star \right\rangle, \\ \langle\langle \Phi_1, \Phi_2 \rangle\rangle_R &= \frac{1}{2}\text{tr}(\sigma_3\sigma_{i_1}\sigma_{i_2})\left\langle Y_{-2} \phi_1(0)^\star \phi_2(0) \right\rangle. \end{aligned} \quad (\text{A.12})$$

Now transform the left handed vertex with $I(z)$ and note

$$\star\star = (-1)^F. \quad (\text{A.13})$$

Then

$$\langle\langle \Phi_1, \Phi_2 \rangle\rangle = (-1)^F \langle\langle \Phi_1, \Phi_2 \rangle\rangle_R. \quad (\text{A.14})$$

So the 2-vertex differs by a sign in the GSO(-) sector between the two star product conventions. This sign plays an important role in fixing the string field reality condition.

Reality conjugation: To formulate the string field reality condition, we need to define reality conjugation. To do this it is helpful to express the string field in the operator formalism. We can define Hermitian and BPZ conjugation of a state $|\Psi\rangle$ or dual state $\langle\Psi|$ using the following rules:

$$\begin{aligned} \left[\mathcal{O}|0\rangle \right]^\dagger &= \langle 0|\mathcal{O}^\dagger, & \left[\langle 0|\mathcal{O} \right]^\dagger &= \mathcal{O}^\dagger|0\rangle, \\ \left[\mathcal{O}|0\rangle \right]^\star &= \langle 0|\mathcal{O}^\star, & \left[\langle 0|\mathcal{O} \right]^\star &= \mathcal{O}^\star|0\rangle, \end{aligned} \quad (\text{A.15})$$

and¹⁴

$$\begin{aligned} (\mathcal{O}_1\mathcal{O}_2)^\dagger &= \mathcal{O}_2^\dagger\mathcal{O}_1^\dagger, & (\mathcal{O}_1\mathcal{O}_2)^\star &= (-1)^{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_2^\star\mathcal{O}_1^\star, \\ (a\mathcal{O}_1 + b\mathcal{O}_2)^\dagger &= a^\star\mathcal{O}_1^\dagger + b^\star\mathcal{O}_2^\dagger, & (a\mathcal{O}_1 + b\mathcal{O}_2)^\star &= a\mathcal{O}_1^\star + b\mathcal{O}_2^\star, \quad a, b \in \mathbb{C} \end{aligned}$$

and

$$\phi(z)^\dagger = \frac{(-1)^\nu}{\bar{z}^{(2h)}}\phi\left(\frac{1}{\bar{z}}\right), \quad \phi(z)^\star = \frac{1}{z^{(2h)}}\phi\left(-\frac{1}{z}\right). \quad (\text{A.16})$$

Here \mathcal{O} are general CFT operators, a, b are complex constants and $\phi(z)$ is a primary of weight h . The sign $(-1)^\nu$ is needed to distinguish between Hermitian and antihermitian fields. We will take the β ghost to be antihermitian, so that the γ ghost and the worldsheet supercurrent are Hermitian. Also, we define BPZ conjugation to leave internal CP factors invariant, whereas Hermitian conjugation takes their conjugate transpose. With these definitions we define *reality conjugation* of a string field Ψ as

$$\Psi^\ddagger = \Psi^{\dagger\star}. \quad (\text{A.17})$$

Note that the order matters: first we perform Hermitian and *then* BPZ conjugation to compute the reality conjugate. In particular,

$$\dagger^\star = \star^\dagger(-1)^F. \quad (\text{A.18})$$

Since $\star^\star = (-1)^F$ this implies that reality conjugation is idempotent,

$$\ddagger^\ddagger = 1, \quad (\text{A.19})$$

and therefore analogous to complex conjugation. Reality conjugation satisfies the important properties

$$(\Psi\Phi)^\ddagger = \Phi^\ddagger\Psi^\ddagger, \quad (a\Psi + b\Phi)^\ddagger = a^\star\Psi^\ddagger + b^\star\Phi^\ddagger, \quad a, b \in \mathbb{C}, \quad (\text{A.20})$$

¹⁴We use the six pointed star * to denote complex conjugation and the five pointed star † to denote BPZ conjugation, hopefully without confusion.

and

$$\langle\langle\Psi\rangle\rangle^* = \langle\langle\Psi^\dagger\rangle\rangle. \quad (\text{A.21})$$

Equations (A.17)-(A.21) hold with the appropriate CP factors attached to the string field.

Twist conjugation: We define *twist conjugation*

$$\Psi^\S = e^{i\pi N} \Psi, \quad (\text{A.22})$$

where N is the number operator (the zero momentum component of L_0). For the bosonic string, twist conjugation is related to the twist operator Ω of [27, 38] by a sign:

$$\Psi^\S = -\Omega\Psi. \quad (\text{A.23})$$

Twist conjugation satisfies

$$\begin{aligned} (\Psi\Phi)^\S &= (-1)^{E(\Psi)E(\Phi)+F(\Psi)F(\Phi)}\Phi^\S\Psi^\S, \\ (a\Psi + b\Phi)^\S &= a\Psi^\S + b\Phi^\S \quad a, b \in \mathbb{C}, \\ \S\S &= (-1)^F. \end{aligned} \quad (\text{A.24})$$

and

$$\langle\langle\Psi^\S\rangle\rangle = \langle\langle\Psi\rangle\rangle. \quad (\text{A.25})$$

Twist conjugation gives a way to map between string fields in the left and right-handed star product conventions[19]. Suppose Φ is a string field in a theory with left handed star product. The equivalent field in the right handed theory is

$$\Phi' = \Phi^\S. \quad (\text{A.26})$$

In the right handed convention, the string field reality condition is[6]:

$$(\Psi')^\dagger = (-1)^F \Psi'. \quad (\text{A.27})$$

Using (A.26) it follows that the reality condition for the left handed theory is

$$\Psi^\dagger = \Psi. \quad (\text{A.28})$$

Note that this means that real string fields in the two conventions differ by a factor of i in the GSO(-) sector. This factor of i corrects the sign discrepancy between the 2-vertices, so the tachyon field has the correct sign kinetic term in either convention.

B Superconformal Generator in the Sliver Frame

The string field G is closely related to the superconformal generator $G_{-1/2}$ in the sliver conformal frame:

$$\mathcal{G} = f_S^{-1} \circ G_{-1/2} = \oint \frac{d\xi}{2\pi i} \left(\sqrt{\frac{\pi}{2}} \sqrt{1 + \xi^2} \right) G(\xi). \quad (\text{B.1})$$

Since this operator is crucial to the construction of the half-brane solution, it is worth understanding in more detail.

Consider an operator of the form

$$\phi[f] = \oint \frac{d\xi}{2\pi i} f(\xi) \phi(\xi), \quad (\text{B.2})$$

where ϕ is a primary of weight h , $f(\xi)$ is a function, and the contour passes inside an annulus of analyticity of $f(\xi)$ around the unit circle. We define Hermitian, BPZ, and dual conjugation[24] of this operator, respectively,

$$\begin{aligned} \phi[f]^\dagger &= \phi[f^\dagger], & f^\dagger(\xi) &= (-1)^\nu \xi^{(2h-2)} f^*(\xi^{-1}), \\ \phi[f]^* &= \phi[f^*], & f^*(\xi) &= -(-\xi)^{(2h-2)} f(-\xi^{-1}), \\ \tilde{\phi}[f] &= \phi[\tilde{f}], & \tilde{f}(\xi) &= \epsilon(\xi) f(\xi). \end{aligned} \quad (\text{B.3})$$

Here $\epsilon(\xi)$ is the step function¹⁵

$$\epsilon(\xi) = \begin{cases} 1 & \text{for } \text{Re}(\xi) > 0 \\ -1 & \text{for } \text{Re}(\xi) < 0 \end{cases}. \quad (\text{B.4})$$

We also define the combinations

$$\begin{aligned} \phi[f]^+ &= \phi[f] + \phi[f]^*, & \phi[f]^- &= \phi[f] - \phi[f]^*, \\ \phi[f]_L &= \frac{1}{2}(\phi[f] + \tilde{\phi}[f]), & \phi[f]_R &= \frac{1}{2}(\phi[f] - \tilde{\phi}[f]). \end{aligned} \quad (\text{B.5})$$

The subscripts L and R denote the left and right halves of the charge $\phi[f]$. In some cases, the action of $\phi[f]_L$ and $\phi[f]_R$ on a state can be described by left or right star multiplication with the appropriate string field. When this is possible, we say that $\phi[f]$ has a *non-anomalous* left/right decomposition.

¹⁵ $\epsilon(\xi)$ has a branch cut extending across the entire imaginary axis. To define dual conjugation carefully, one should represent $\epsilon(\xi)$ as the limit of a sequence of functions which are analytic in some (vanishingly thin) annulus containing the unit circle[24].

Consider the operators

$$\begin{aligned}\mathcal{L} &= \mathbf{T}[\ell], & \ell(\xi) &= (1 + \xi^2) \tan^{-1} \xi, \\ \mathcal{G} &= \mathbf{G}[g], & g(\xi) &= \sqrt{\frac{\pi}{2}} \sqrt{1 + \xi^2}.\end{aligned}\tag{B.6}$$

where

$$\mathbf{T}[v] = \oint \frac{d\xi}{2\pi i} v(\xi) T(\xi), \quad \mathbf{G}[s] = \oint \frac{d\xi}{2\pi i} s(\xi) G(\xi).\tag{B.7}$$

The first is the familiar \mathcal{L}_0 of Schnabl[21], and the second is the operator $G_{-1/2}$ in the sliver conformal frame. The functions $\ell(\xi)$ and $g(\xi)$ have branch points at $+i$ and $-i$, connected by a branch cut on the imaginary axis passing through infinity. The branch points of $\ell(\xi)$ takes the form $x \ln x$ for small $x = \xi \pm i$, whereas those of $g(\xi)$ take the form \sqrt{x} . The BPZ conjugate operators are

$$\begin{aligned}\mathcal{L}^* &= \mathbf{T}[\ell^*], & \ell^*(\xi) &= (1 + \xi^2) \tan^{-1} \frac{1}{\xi}, \\ \mathcal{G}^* &= \mathbf{G}[g^*], & g^*(\xi) &= \sqrt{\frac{\pi}{2}} \xi \sqrt{1 + \frac{1}{\xi^2}}.\end{aligned}\tag{B.8}$$

ℓ^*, g^* also have branch points at $\pm i$, but the cuts now extend on the imaginary axis through the origin. Note that by factoring ξ into the square root in (B.8), g^* formally appears to be the same as g . In fact, they are equal up to a sign:

$$g^*(\xi) = \epsilon(\xi) g(\xi).\tag{B.9}$$

This means that the BPZ conjugate of \mathcal{G} is equal to its dual conjugate:

$$\mathcal{G}^* = \tilde{\mathcal{G}}.\tag{B.10}$$

It is also useful to consider the operators

$$\begin{aligned}\mathcal{L}^+ &= \mathcal{L} + \mathcal{L}^* = \mathbf{T}[\ell^+], & \ell^+(\xi) &= \frac{\pi}{2} (1 + \xi^2) \epsilon(\xi), \\ \tilde{\mathcal{L}}^+ &= \mathbf{T}[\tilde{\ell}^+], & \tilde{\ell}^+(\xi) &= \frac{\pi}{2} (1 + \xi^2).\end{aligned}\tag{B.11}$$

The Hermitian conjugates of $\mathcal{L}, \mathcal{L}^*$ and \mathcal{G} are equal to their BPZ conjugates. For \mathcal{G}^* and $\tilde{\mathcal{L}}^+$ there is a sign difference: $\mathcal{G}^{*\dagger} = \mathcal{G} = -\mathcal{G}^{**}$ and $\tilde{\mathcal{L}}^{+\dagger} = \tilde{\mathcal{L}}^+ = -\tilde{\mathcal{L}}^{**}$.

The string fields K and G can be defined through the action of $\mathcal{L}^+, \mathcal{G}$, and their dual conjugates on a test state:

$$\begin{aligned}\mathcal{L}^+ \Phi &= K \Phi + \Phi K, & \tilde{\mathcal{L}}^+ \Phi &= K \Phi - \Phi K = \partial \Phi, \\ \sigma_1 \mathcal{G}^* \Phi &= G \Phi + (-1)^{F(\Phi)} \Phi G, & \sigma_1 \mathcal{G} \Phi &= G \Phi - (-1)^{F(\Phi)} \Phi G = \delta \Phi.\end{aligned}\tag{B.12}$$

This definition implies three consistency conditions:

$$\begin{aligned}
\mathbf{1)} \quad & \tilde{\mathcal{L}}^+|I\rangle = 0 \quad \mathcal{G}|I\rangle = 0 \\
\mathbf{2)} \quad & \mathcal{L}_L^+(\Phi\Psi) = (\mathcal{L}_L^+\Phi)\Psi \quad \mathcal{L}_R^+(\Phi\Psi) = \Phi(\mathcal{L}_R^+\Psi) \\
& \mathcal{G}_L(\Phi\Psi) = (\mathcal{G}_L\Phi)\Psi \quad \mathcal{G}_R(\Phi\Psi) = (-1)^{\epsilon(\Phi)}\Phi(\mathcal{G}_R\Psi) \\
\mathbf{3)} \quad & \{\mathcal{G}_L, \mathcal{G}_R\} = 0, \quad [\mathcal{G}_L, \mathcal{L}_R^+] = 0, \quad [\mathcal{L}_L^+, \mathcal{L}_R^+] = 0
\end{aligned} \tag{B.13}$$

The first condition follows from setting $\Phi = |I\rangle$ in (B.12). The second follows from associativity of the star product. The third condition follows from the assumption that $\mathcal{L}^+, \mathcal{G}$, and their dual conjugates should have well defined action on K and G ; in other words, K and G can be consistently star multiplied among themselves. If these three conditions are satisfied, \mathcal{L}^+ and \mathcal{G} have a non-anomalous left/right decomposition.

It is not difficult to verify conditions **1)** and **2)** by contracting with “reasonable” test states (for example, wedge states of positive width with insertions placed away from the midpoint) and mapping to the upper half plane. Condition **3)** is more subtle and is worth checking explicitly. We can compute the commutators using the superconformal algebra expressed in the form

$$\begin{aligned}
\left[\mathbf{T}[v_1], \mathbf{T}[v_2] \right] &= \mathbf{T}[v_2\partial v_1 - v_1\partial v_2], \\
\left[\mathbf{T}[v], \mathbf{G}[s] \right] &= \mathbf{G}\left[\frac{1}{2}s\partial v - v\partial s\right], \\
\left\{ \mathbf{G}[s_1], \mathbf{G}[s_2] \right\} &= \mathbf{T}[2s_1s_2].
\end{aligned} \tag{B.14}$$

To be careful about singularities at the midpoint, we regulate \mathcal{G} and \mathcal{L}^+ by replacing

$$\begin{aligned}
g(\xi) &\rightarrow g(\lambda\xi), & g(\xi)^* &\rightarrow g(\xi/\lambda)^*, \\
\ell(\xi) &\rightarrow \ell(\lambda\xi), & \ell(\xi)^* &\rightarrow \ell(\xi/\lambda)^*.
\end{aligned} \tag{B.15}$$

Condition **3)** is satisfied in the limit $\lambda \rightarrow 1^-$. Another check is to expand the operators

in modes

$$\begin{aligned}
\mathcal{L} &= L_0 + \frac{2}{3}L_2 - \frac{2}{15}L_4 + \dots = L_0 + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} L_{2n}, \\
\mathcal{L}^* &= L_0 + \frac{2}{3}L_{-2} - \frac{2}{15}L_{-4} + \dots = L_0 + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} L_{-2n}, \\
\tilde{\mathcal{L}}^+ &= \frac{\pi}{2}(L_1 + L_{-1}), \\
\mathcal{G} &= \sqrt{\frac{\pi}{2}} \left(G_{-1/2} + \frac{1}{2}G_{3/2} - \frac{1}{8}G_{7/2} + \dots \right) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \binom{1/2}{n} G_{2n-\frac{1}{2}}, \\
\mathcal{G}^* &= \sqrt{\frac{\pi}{2}} \left(G_{1/2} + \frac{1}{2}G_{-3/2} - \frac{1}{8}G_{-7/2} + \dots \right) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \binom{1/2}{n} G_{\frac{1}{2}-2n}, \quad (\text{B.16})
\end{aligned}$$

and calculate using the usual mode commutators of the superconformal algebra. Again we have found that **3)** is satisfied, and the infinite sums needed in the computation are absolutely convergent.¹⁶ Given these and other checks, we believe that $\mathcal{L}^+, \mathcal{G}$ have a non-anomalous left/right decomposition.

The operators $\mathcal{L}, \mathcal{L}^*, \tilde{\mathcal{L}}^+$ and $\mathcal{G}, \mathcal{G}^*$ form a super-Lie algebra with commutators,

$$\begin{aligned}
[\mathcal{L}, \mathcal{L}^*] &= \mathcal{L}^+, & [\mathcal{L}, \tilde{\mathcal{L}}^+] &= \tilde{\mathcal{L}}^+, & [\mathcal{L}^*, \tilde{\mathcal{L}}^+] &= -\tilde{\mathcal{L}}^+, \\
\{\mathcal{G}, \mathcal{G}^*\} &= 2\mathcal{L}^+, & \{\mathcal{G}, \mathcal{G}\} &= 2\tilde{\mathcal{L}}^+, & \{\mathcal{G}^*, \mathcal{G}^*\} &= 2\tilde{\mathcal{L}}^+, \\
[\mathcal{L}, \mathcal{G}] &= \frac{1}{2}\mathcal{G}, & [\mathcal{L}^*, \mathcal{G}] &= -\frac{1}{2}\mathcal{G}, & [\tilde{\mathcal{L}}^+, \mathcal{G}] &= 0, \\
[\mathcal{L}, \mathcal{G}^*] &= \frac{1}{2}\mathcal{G}^*, & [\mathcal{L}^*, \mathcal{G}^*] &= -\frac{1}{2}\mathcal{G}^*, & [\tilde{\mathcal{L}}^+, \mathcal{G}^*] &= 0. \quad (\text{B.17})
\end{aligned}$$

This can be thought of as a supersymmetric extension of the special projector algebra[21, 24]. Assuming (B.12), this algebra can be compactly summarized by the relations

$$G^2 = K \quad [K, G] = 0 \quad \frac{1}{2}\mathcal{L}^- K = K \quad \frac{1}{2}\mathcal{L}^- G = \frac{1}{2}G \quad (\text{B.18})$$

C Splitting Charges and Midpoint Insertions

When computing the action and gauge invariant overlap, we implicitly assumed cyclicity of the vertices $\langle\langle \cdot \rangle\rangle$ and $\langle\langle \cdot \rangle\rangle_{\mathcal{V}}$. However, the presence of midpoint insertions makes this

¹⁶One further subtlety is that the vanishing of left/right commutators is not always sufficient to guarantee that nonpolynomial combinations of left and right charges commute. Splitting \mathcal{L} into left and right halves we find $[\mathcal{L}_L, \mathcal{L}_R] = 0$, but \mathcal{L}_L actually does not commute with $e^{-s\mathcal{L}_R}$. This is crucial for recovering closed string moduli in Schnabl gauge amplitudes[39, 40]. We have found no evidence for similar anomalies when splitting \mathcal{L}^+ and \mathcal{G} .

subtle. In the K, B, c, G subalgebra, cyclicity of $\langle\langle \cdot \rangle\rangle$ and $\langle\langle \cdot \rangle\rangle_{\mathcal{V}}$ requires

$$[K, Y_{-2}] = 0, \quad [K, \mathcal{V}] = 0, \quad (\text{C.1})$$

and likewise for B and G . (The cyclicity of c appears unproblematic since the c insertion is far from the midpoint.) While the geometry of the Witten vertex appears to guarantee that midpoint insertions commute, this expectation fails in at least some examples[41].¹⁷

To keep the discussion general, consider a string field Φ corresponding to a vertical line integral insertion of a primary $\phi(z)$ of weight $h > 0$ in the cylinder coordinate frame:

$$\Phi \rightarrow \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \phi(z). \quad (\text{C.2})$$

Explicitly we can write

$$\Phi = \phi_L |I\rangle, \quad \text{where} \quad \phi_L = \int_L \frac{d\xi}{2\pi i} \left(\sqrt{\frac{\pi}{2}} \sqrt{1 + \xi^2} \right)^{2(h-1)} \phi(\xi), \quad (\text{C.3})$$

and the contour L is over the positive half of the unit circle. Let $m = m(i)|I\rangle$ correspond to an insertion of a dimension zero primary $m(z)$ at the midpoint. Then we can show that $[\Phi, m] = 0$ if and only if

$$\lim_{\sigma \rightarrow \frac{\pi}{2}} [\phi_L, m(e^{i\sigma})] = 0. \quad (\text{C.4})$$

Suppose $\phi(z)$ and $m(z)$ have an OPE of the form,

$$\phi(z+w)m(z) \sim \sum_{n=1}^{\infty} \frac{1}{w^n} V_n(z), \quad (\text{C.5})$$

where $V_n(z)$ are local operators of dimension $h - n$. Computing the commutator (C.4) we can prove the following:

Claim. *The limit of the commutator (C.4) vanishes if and only if one of the two following criteria are satisfied:*

- a) *If $h \in \mathbb{Z} + \frac{1}{2}$, then $V_n(z) = 0$ for all $n > h$.*
- b) *If $h \in \mathbb{Z}$, then $V_n(z) = 0$ for all n in the range $2h > n \geq h$.*

¹⁷A related question is whether the derivations \mathcal{L}^- and \mathcal{B}^- annihilate $\langle\langle \cdot \rangle\rangle$ and $\langle\langle \cdot \rangle\rangle_{\mathcal{V}}$. This can be shown along similar lines to the argument presented here.

In the current context, the role of Φ is played by K, B , and G and the role of m is played by Y_{-2} and \mathcal{V} . According to the above claim, K, B and G commute with Y_{-2} and \mathcal{V} if and only if the OPEs between T, G, b , and Y_{-2}, \mathcal{V} take the following form:

$$\begin{aligned} T(z+w)Y_{-2}(z, \bar{z}) &\sim \mathcal{O}(w^{-1}), & T(z+w)\mathcal{V}(z, \bar{z}) &\sim \mathcal{O}(w^{-1}), \\ G(z+w)Y_{-2}(z, \bar{z}) &\sim \mathcal{O}(w^{-1}), & G(z+w)\mathcal{V}(z, \bar{z}) &\sim \mathcal{O}(w^{-1}), \\ b(z+w)Y_{-2}(z, \bar{z}) &\sim \mathcal{O}(w^{-1}), & b(z+w)\mathcal{V}(z, \bar{z}) &\sim \mathcal{O}(w^{-1}), \end{aligned} \quad (\text{C.6})$$

and likewise for the antiholomorphic currents $\tilde{T}, \tilde{G}, \tilde{b}$. Let us assume that Y_{-2} and \mathcal{V} take the explicit forms given in (1.4) and (5.2). The OPEs with T follow from the fact that Y_{-2} and \mathcal{V} are dimension $(0,0)$ primaries. The OPEs with G follow from the fact that Y_{-2} and \mathcal{V} are superconformal primaries. Finally the OPEs with b follow from the fact that in the bc CFT Y_{-2} and \mathcal{V} are proportional to $c\tilde{c}$, which produces only a single pole in the OPE with b . Therefore the vertices $\langle\langle \cdot \rangle\rangle$ and $\langle\langle \cdot \rangle\rangle_{\mathcal{V}}$ are expected to be cyclic when evaluated on fields in the K, B, c, G subalgebra.

D Phantom Piece and Super-Wedge States

In this appendix we prove that the phantom term (4.6) can be described by a super-wedge state (4.30) in the large N limit. First we give an explicit definition of super-wedge states in the Fock space. Write

$$e^{-\alpha K} e^{i\beta G} = f_1(\alpha, \beta) + iG f_2(\alpha, \beta), \quad (\text{D.1})$$

where

$$\begin{aligned} f_1(\alpha, \beta) &= \Omega^\alpha \cos(\beta\sqrt{K}), \\ f_2(\alpha, \beta) &= \Omega^\alpha \frac{\sin(\beta\sqrt{K})}{\sqrt{K}}. \end{aligned} \quad (\text{D.2})$$

We can compute the Fock space coefficients of (f_1, f_2) using the linear functional (4.39):

$$\begin{aligned} L_{f_1}(x^h) &= \frac{1}{(h-1)!} \int_0^\infty dt t^{h-1} \cos(\beta\sqrt{t}) e^{-(\alpha+1)t} \\ &= \frac{1}{(1+\alpha)^h} {}_1F_1 \left[h, \frac{1}{2}, -\frac{\beta^2}{4(1+\alpha)} \right], \end{aligned} \quad (\text{D.3})$$

$$\begin{aligned} L_{f_2}(x^h) &= \frac{1}{(h-1)!} \int_0^\infty dt t^{h-1} \frac{\sin(\beta\sqrt{t})}{\sqrt{t}} e^{-(\alpha+1)t} \\ &= \frac{\beta}{(1+\alpha)^h} {}_1F_1 \left[h, \frac{3}{2}, -\frac{\beta^2}{4(1+\alpha)} \right], \end{aligned} \quad (\text{D.4})$$

where ${}_1F_1$ is the confluent hypergeometric function.

Consider the states (X_N, Y_N) appearing in the phantom piece through equation (4.8). We can also define these states using the linear functional (4.39):

$$\begin{aligned} L_{X_N}(x^h) &= \frac{1}{(h-1)!} \int_0^\infty dt t^{h-1} \frac{(1+ia\sqrt{t})^N + (1-ia\sqrt{t})^N}{2} e^{-(N+1)t}, \\ L_{Y_N}(x^h) &= \frac{1}{(h-1)!} \int_0^\infty dt t^{h-1} \frac{(1+ia\sqrt{t})^N - (1-ia\sqrt{t})^N}{2iaN\sqrt{t}} e^{-(N+1)t}. \end{aligned} \quad (\text{D.5})$$

To compute the large N limit, substitute $s = (N+1)t$ in the integrand so that for example

$$L_{X_N}(x^h) = \frac{1}{2(h-1)!} \frac{1}{(N+1)^h} \int_0^\infty ds s^{h-1} \left[\left(1 + ia\sqrt{\frac{s}{N+1}}\right)^N + \left(1 - ia\sqrt{\frac{s}{N+1}}\right)^N \right] e^{-s}. \quad (\text{D.6})$$

Now approximate

$$\begin{aligned} \left(1 \pm ia\sqrt{\frac{s}{N+1}}\right)^N &= \exp \left[N \ln \left(1 \pm ia\sqrt{\frac{s}{N+1}}\right) \right] \\ &= \exp \left[N \left(\pm ia\sqrt{\frac{s}{N}} + \frac{a^2}{2} \frac{s}{N} + \mathcal{O}(N^{-1/2}) \dots \right) \right] \\ &= e^{\pm ia\sqrt{Ns}} e^{a^2 s/2} [1 + \mathcal{O}(N^{-1/2})], \end{aligned} \quad (\text{D.7})$$

so that

$$\begin{aligned} L_{X_N}(x^h) &= \frac{1}{(h-1)!} \frac{1}{N^h} \int_0^\infty ds s^{h-1} \cos(a\sqrt{Ns}) e^{-(1-\frac{a^2}{2})s} [1 + \mathcal{O}(N^{-1/2})] \\ &= \frac{1}{N^h} \left(\frac{2}{2-a^2} \right)^h {}_1F_1 \left(h, \frac{1}{2}, -\frac{1}{4} \frac{2}{2-a^2} a^2 N \right) [1 + \mathcal{O}(N^{-1/2})]. \end{aligned} \quad (\text{D.8})$$

Similarly,

$$L_{Y_N}(x^h) = \frac{1}{N^h} \left(\frac{2}{2-a^2} \right)^h {}_1F_1 \left(h, \frac{3}{2}, -\frac{1}{4} \frac{2}{2-a^2} a^2 N \right) [1 + \mathcal{O}(N^{-1/2})]. \quad (\text{D.9})$$

Comparing with equations (D.3) and (D.4), this precisely corresponds to the large N behavior of the super-wedge state $e^{iNaG} \Omega^{N(1-\frac{a^2}{2})}$, as claimed in equation (4.31).

To simplify the large N limit further we use the asymptotic formula

$${}_1F_1(a, b, z) = \frac{\Gamma(a)}{\Gamma(b-a)} e^{i\pi a} \frac{1}{z^a} [1 + \mathcal{O}(z^{-1})], \quad (\text{large } |z|, \quad \text{Re}(z) < 0). \quad (\text{D.10})$$

Thus,

$$L_{X_N}(x^h) = \frac{2(-1)^h (2h-1)!}{(aN)^{2h} (h-1)!} [1 + \mathcal{O}(N^{-1/2})], \quad (\text{D.11})$$

$$L_{Y_N}(x^h) = \frac{4(-1)^h (2h-3)!}{(aN)^{2h} (h-2)!} [1 + \mathcal{O}(N^{-1/2})]. \quad (\text{D.12})$$

This agrees with the large N behavior of the sums quoted in (4.29). We have verified this behavior numerically.

E Details of Energy Computation

In this appendix we give some details of the computation of the action for the simple half-brane solution. To avoid cluttered formulas, it is helpful to introduce the notation,

$$(\Phi_1, \Phi_2) = \left\langle\left\langle \Phi_1 \frac{1}{1+K} \Phi_2 \frac{1}{1+K} \right\rangle\right\rangle. \quad (\text{E.1})$$

The kinetic term of the action can be expressed as the sum of two terms:

$$\langle\langle \Psi Q \Psi \rangle\rangle = -(1) + (2), \quad (\text{E.2})$$

where

$$(1) = (cGBc, Q(cGBc)), \quad (2) = (cGBcG, Q(cGBc)G). \quad (\text{E.3})$$

Now replace the G insertions with supersymmetry variations δ acting inside the vertex, following (5.34). This generates many terms, some of which vanish by ϕ -momentum conservation or by \mathcal{L}^- or \mathcal{B}^- invariance of the vertex. In the end the answer simplifies to

$$(1) = -(cK, \gamma^2) + 5(B\gamma^2, c\partial c) + 2(\gamma, \partial\gamma c) - 4(cB\gamma, \partial\gamma c) - 4(cB\gamma, \gamma Kc), \quad (\text{E.4})$$

and

$$\begin{aligned} (2) = & -(cK, \gamma^2 K) - 4(cB\gamma, c\partial\gamma K) + 2(cB\gamma, \partial c\gamma K) + (B\gamma^2, Kc\partial c) - 2(cB\gamma, K\partial\gamma c) \\ & + 4(cB\gamma, K\gamma Kc) + (cB\gamma, \partial\gamma\partial c) + 2(cB\gamma, \partial^2\gamma c) - (cB\gamma, \gamma\partial^2 c) - \frac{1}{2}(B\gamma^2, c\partial^2 c). \end{aligned} \quad (\text{E.5})$$

We compute the inner products $(,)$ by mapping them to the appropriate correlation function on the cylinder, evaluating the correlator with (5.35), and performing the Schwinger

integrals. For (1) the inner products turn out to be

$$\begin{aligned}(cK, \gamma^2) &= \frac{2}{\pi^2}, & (B\gamma^2, c\partial c) &= \frac{1}{\pi^2}, & (\gamma, \partial\gamma c) &= -\frac{2}{\pi^2}, \\ (cB\gamma, \partial\gamma c) &= -\frac{1}{\pi^2}, & (cB\gamma, \gamma Kc) &= \frac{6}{\pi^4},\end{aligned}\tag{E.6}$$

giving

$$(1) = -\frac{2}{\pi^2} + \frac{5}{\pi^2} - \frac{4}{\pi^2} + \frac{4}{\pi^2} - \frac{24}{\pi^4} = \frac{3}{\pi^2} - \frac{24}{\pi^4}.\tag{E.7}$$

For (2) we have the inner products

$$\begin{aligned}(cK, \gamma^2 K) &= -\frac{1}{\pi^2}, & (cB\gamma, c\partial\gamma K) &= -\frac{1}{\pi^2}, & (cB\gamma, \partial c\gamma K) &= -\frac{1}{2\pi^2}, \\ (B\gamma^2, Kc\partial c) &= -\frac{1}{2\pi^2}, & (cB\gamma, K\partial\gamma c) &= \frac{1}{\pi^2}, & (cB\gamma, K\gamma Kc) &= \frac{1}{2\pi^2} - \frac{6}{\pi^4}, \\ (cB\gamma, \partial\gamma\partial c) &= -\frac{1}{\pi^2}, & (cB\gamma, \partial^2\gamma c) &= \frac{1}{\pi^2}, & (cB\gamma, \gamma\partial^2 c) &= (B\gamma^2, c\partial^2 c) = 0,\end{aligned}\tag{E.8}$$

giving

$$\begin{aligned}(2) &= \frac{1}{\pi^2} + \frac{4}{\pi^2} - \frac{1}{\pi^2} - \frac{1}{2\pi^2} - \frac{2}{\pi^2} + \frac{2}{\pi^2} - \frac{24}{\pi^4} - \frac{1}{\pi^2} + \frac{2}{\pi^2} + 0 + 0 \\ &= \frac{5}{\pi^2} - \frac{1}{2\pi^2} - \frac{24}{\pi^4}.\end{aligned}\tag{E.9}$$

Adding things up

$$\begin{aligned}\langle\langle\Psi Q\Psi\rangle\rangle &= -(1) + (2) \\ &= -\frac{3}{\pi^2} + \frac{24}{\pi^4} + \frac{5}{\pi^2} - \frac{1}{2\pi^2} - \frac{24}{\pi^4} \\ &= \frac{3}{2\pi^2}.\end{aligned}\tag{E.10}$$

The energy is

$$E = -\frac{1}{6}\langle\langle\Psi Q\Psi\rangle\rangle = -\frac{1}{4\pi^2},\tag{E.11}$$

which is precisely $-1/2$ times the tension of the D-brane.

F Auxiliary Tachyon Coefficient

In this appendix we compute the coefficient of the auxiliary tachyon state $c_1|0\rangle$ for the Schnabl-like half-brane solution in the L_0 level expansion. To achieve this we write the

Schnabl-like solution in the form

$$\begin{aligned}
\Psi_{\text{Sch}} &= - \sum_{n=0}^{\infty} \psi'_n + \Gamma \\
&= - \sum_{n=0}^{\infty} \sum_{0 \leq k \leq n/2} \binom{n}{2k} a^{2k} \frac{d^{k+1}}{dr^{k+1}} \Big|_{r=0} cB\Omega^{n+r} c(1 + iaG)\Omega \\
&\quad - i \sum_{n=0}^{\infty} \sum_{0 \leq k \leq \frac{n-1}{2}} \binom{n}{2k+1} a^{2k+1} \frac{d^{k+1}}{dr^{k+1}} \Big|_{r=0} cBG\Omega^{n+r} c(1 + iaG)\Omega \\
&\quad + B\gamma^2(1 + iaG)\Omega.
\end{aligned} \tag{F.1}$$

We can drop the phantom term since it vanishes in the Fock space. The states inside the sums can be expressed using the operator formalism of Schnabl[21, 42], which yields an expression for the solution in terms of a canonically ordered set of mode operators acting on the $SL(2, \mathbb{R})$ vacuum. Using (4.29) one can argue that the infinite sums above converge for any coefficient in the Fock space as long as the parameter a is restricted to the range $-\sqrt{2} \leq a \leq \sqrt{2}$.

Expanding (F.1) in the Fock space we can extract the coefficient of the auxiliary tachyon. Define two functions

$$\begin{aligned}
\phi_1(r) &= \frac{1}{\pi X^2} \left[\frac{1}{\pi} \cos^2 \left(\frac{\pi}{2} X_+ \right) \sin(\pi X_-) - \frac{1}{\pi} \sin(\pi X_+) \cos^2 \left(\frac{\pi}{2} X_- \right) \right. \\
&\quad \left. - (X_+ - 1) \cos^2 \left(\frac{\pi}{2} X_- \right) + (X_- + 1) \cos^2 \left(\frac{\pi}{2} X_+ \right) \right], \\
\phi_2(r) &= -\frac{d}{dr} \phi_1(r) + \frac{1}{X} \left[-\frac{1}{2\pi} \sin(\pi X_+) \cos \left(\frac{\pi}{2} X_- \right) - \frac{1}{2} (X_+ - 1) \cos \left(\frac{\pi}{2} X_- \right) \right. \\
&\quad \left. - \frac{1}{4\pi} \sin(\pi X_+) \sin(\pi X_-) - \frac{1}{2\pi} \cos^2 \left(\frac{\pi}{2} X_+ \right) (\cos(\pi X_-) + 1) \right. \\
&\quad \left. - \frac{1}{4} (X_+ - 1) \sin(\pi X_-) \right],
\end{aligned} \tag{F.2}$$

where for short we have denoted

$$X = \frac{2}{r+2}, \quad X_+ = \frac{r+1}{r+2}, \quad X_- = \frac{-r+1}{r+2}. \tag{F.3}$$

The auxiliary coefficient is then

$$\phi = \sum_{n=0}^{\infty} \left[- \sum_{0 \leq k \leq n/2} \binom{n}{2k} a^{2k} \frac{d^{k+1} \phi_1(n)}{dn^{k+1}} + \sum_{0 \leq k \leq \frac{n-1}{2}} \binom{n}{2k+1} a^{2k+2} \frac{d^{k+1} \phi_2(n)}{dn^{k+1}} \right]. \tag{F.4}$$

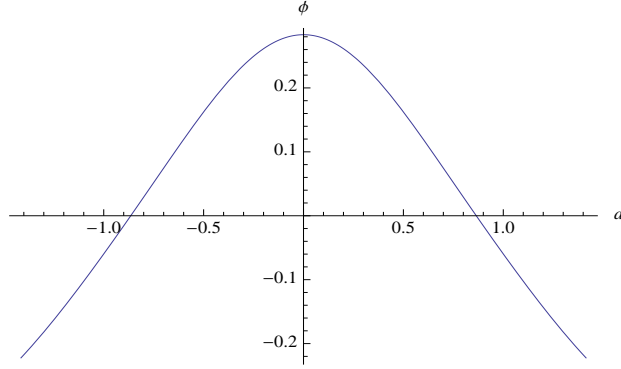


Figure F.1: Coefficient of the auxiliary tachyon $c_1|0\rangle$ in the Schnabl-like solution, as a function of $a \in [-\sqrt{2}, \sqrt{2}]$.

Since ϕ_1 and ϕ_2 vanish as $1/r^3$ for large r , (4.29) implies that the terms in the summand vanish as $1/n^8$ for sufficiently large n . We have checked this behavior numerically. Therefore (F.4) is a convergent sum if $-\sqrt{2} \leq a \leq \sqrt{2}$. Unfortunately, the multiple derivatives of ϕ_1 and ϕ_2 make a direct numerical evaluation of (F.4) very time-consuming. To evaluate (F.4) with sufficient precision, we found it necessary to expand ϕ_1 and ϕ_2 in powers of $\frac{1}{r+2}$ out to $\frac{1}{(r+2)^{40}}$, which simplifies the numerical computation of derivatives. For $a = 1$ we found the auxiliary tachyon coefficient to be

$$\phi = -.0599156. \quad (\text{F.5})$$

More interesting is the plot of the auxiliary tachyon coefficient as a function of a , shown in figure F.1. At $a = 0$ the coefficient corresponds to that of a tachyon vacuum solution, and has positive expectation value, as we would expect from the usual picture of the cubic potential in bosonic string field theory. However, as a becomes large, the expectation value becomes zero and even negative. This suggests that the negative energy of the half-brane solution is not principally due to the condensation of the auxiliary tachyon. This is one way to see that the Schnabl-like solution must not satisfy the reality condition.

We have also computed the coefficients for a few descendents of the auxiliary tachyon. Let us denote coefficients of the states

$$(L_{-2})^n c_1|0\rangle, \quad (L_{-4})^n c_1|0\rangle \quad (\text{F.6})$$

by x_n and y_n respectively for $n \geq 1$. At $a = 1$ we have found the explicit values

$$\begin{aligned} x_1 &= .067747, & y_1 &= -.019133, \\ x_2 &= .0060976, & y_2 &= .000064506, \\ x_3 &= -.000042514, & y_3 &= 7.9488 \times 10^{-7}. \end{aligned} \tag{F.7}$$

We have computed x_n and y_n out to $n = 60$ and found that they decay significantly faster than the corresponding coefficients of $(L_{-2})^n|0\rangle$ and $(L_{-4})^n|0\rangle$ of the sliver state. We therefore believe that the Schnabl-like solution is a regular state in the L_0 level expansion.

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